이학박사 학위논문

Powers of Dehn twists generating right-angled Artin groups

(직교 아틴 군을 생성하는 덴 뒤틀림들의 거듭제곱들)

2020년 2월

서울대학교 대학원

수리과학부

서동균

Powers of Dehn twists generating right-angled Artin groups

(직교 아틴 군을 생성하는 덴 뒤틀림들의 거듭제곱들)

지도교수 강정수

이 논문을 이학박사 학위논문으로 제출함

2019년 10월

서울대학교 대학원

수리과학부

서동균

서동균의 이학박사 학위논문을 인준함

2019년 12월

위 원 장	____________________	(인)
부 위 원 장	____________________	(인)
위 원	____________________	(인)
위 원	____________________	(인)
위 원	____________________	(인)

Powers of Dehn twists generating right-angled Artin groups

A dissertation

submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

to the faculty of the Graduate School of

Seoul National University

by

Donggyun Seo

Dissertation Director : Professor Jungsoo Kang

Department of Mathematical Sciences

Seoul National University

February 2020

Abstract

Powers of Dehn twists generating right-angled Artin groups

Donggyun Seo

Department of Mathematical Sciences
The Graduate School
Seoul National University

The author found right-angled Artin subgroups of mapping class groups, which are generated by powers of Dehn twists. For example, for finitely many simple closed curves of a surface, if the intersection numbers of all pairs do not exceed one, then the seventh powers of their Dehn twists generate a right-angled Artin subgroup of the mapping class group. Throughout the proof, we analyze the dual tree of a simple closed curve and tree actions of fundamental groups of surfaces and find elliptic isometries of dual trees associated with lifts of Dehn twists.

For every closed orientable surface Σ of genus ≥ 2, we showed that there exists a faithful quasi-isometry action of $\mathrm{Aut}(\pi_1(\Sigma))$ on a CAT(0) cube complex such that the restriction of the action to the inner automorphism group of $\mathrm{Aut}(\pi_1(\Sigma))$ is an isometric action. We develop a method to compute lifts of Dehn twists using the pocsets inherited from CAT(0) cube complexes and bridges. (A pocset is a partially ordered set with a complementation, in addition, a bridge is a convex subcomplex, which is studied by Behrstock–Charney [1].)

Key words: right-angled Artin group, mapping class group, Dehn twist, CAT(0) cube complex
Student Number: 2013–22906

Contents

Chapter 1

Introduction

The mapping class group of an orientable finite type surface is the group of isotopy classes of orientation preserving self-homeomorphisms. Dehn, Nielsen and Thurston established the foundation of the study of mapping class groups. From the background, Thurston presented the fact that a mapping torus of a surface is hyperbolic if and only if the attaching map is pseudo-Anosov.

The main object of the first part of this dissertation is right-angled Artin subgroups of mapping class groups. A right-angled Artin group is an Artin group in which every coefficient of its Coxeter matrix is either 2 or ∞. In other words, all relations in the presentation of right-angled Artin groups are commuting relations. ($[a, b] = aba^{-1}b^{-1} = 1$.) Right-angled Artin groups have been developed in geometric group theory since Charney's work.

Back to the point, note that mapping class groups admit a finite presentation with finitely many Dehn twists; see [17, Theorem 5.7]. Wajnryb showed that a mapping class group can be presented by Humphries generator with commuting relations, braid relations, a 3-chain relation, a lantern relation and a hyperelliptic relation. Note that two mapping classes are disjoint from each other if and only if their supports are disjoint. Koberda [24] noticed that sufficiently large powers of mapping class groups does not have relations except for commuting relations. That is, a subgroup generated by large powers of mapping classes is isomorphic to a right-angled Artin group.

Recently, it is evident that there are infinitely many right-angled Artin subgroups of mapping class groups through Koberda's notion. Various relationship between right-angled Artin groups and mapping class groups have been revealed. Clay–Leininger–Mangahas [14] showed that every right-angled Artin group can be quasi-isometrically embedded into some mapping class group. An embedding of a mapping class group into a right-angled Artin group cannot happen in most cases; see Kapovich–Leeb [22, Theorem 4.2] and Koberda [24, Theorem 1.5]. Clay–Leininger–Margalit [13] proved that almost all mapping class groups are commensurable with no right-angled Artin groups. By Kim–Koberda [23], if a surface is of low complexity, then the defining graph of every right-angled Artin subgroup is embedded into the curve graph. By Crisp–Paris [15], second powers of Dehn twists can generate a right-angled Artin subgroup if the intersection numbers of all pairs of them and the defining graph are not too complex. Funar [18] generalized Crisp–Paris' result in a more general setting.

The first result of this dissertation is derived from trials to interpret Koberda's method. We analyze his method in a combinatorial setting. Note that there are well-known combinatorial tools for mapping class groups, for example, curve complexes, arc complexes, train tracks, etc. Masur–Minsky [28, 29] solved several word problems about mapping class groups. Among these objects, We study the dual tree of a simple closed curve and the action of a surface group on the tree, and extend this action to the action of the automorphism group of a surface group.

The theme of the second part of this paper is quasi-isometric actions of automorphism groups of surface groups on CAT(0) cube complexes. The main theorem explain the relationship between an isometric action of a surface group and a quasi-isometric action of its automorphism group on a CAT(0) cube complex. In order to prove the main theorem, we will extend several tools established in Part I and develop new tools for the study of an affine action of CAT(0) cube complexes.

Remark 1.0.1. Note that the argument cannot be applied to $\mathrm{SL}(2,\mathbb{Z})$, the orientation-preserving automorphism group of the torus group, because the Euclidean plane is not hyperbolic.[1] Nevertheless, they are contained in the class of cubical affine groups defined in Chapter 1.2.

Properties introduced in Part II will be used in the future research for square-tiled surfaces.[2] As a future research, the author focuses on a linear representation of the affine group of a square-tiled surface. The kernel of this

[1] In fact, the quasi-isometry group of $\mathbb{Z}^2$ is trivial.

[2] Square-tiled surface have been studied as a kind of translation surfaces; refer to Kontsevich–Zorich [25] and Eskin–Masur [16]. Compact NPC cube complexes obtained from finitely many simple closed curves of surfaces are regarded as square-tiled surfaces and the CAT(0) cube complexes in this dissertation are considered as branched covers of

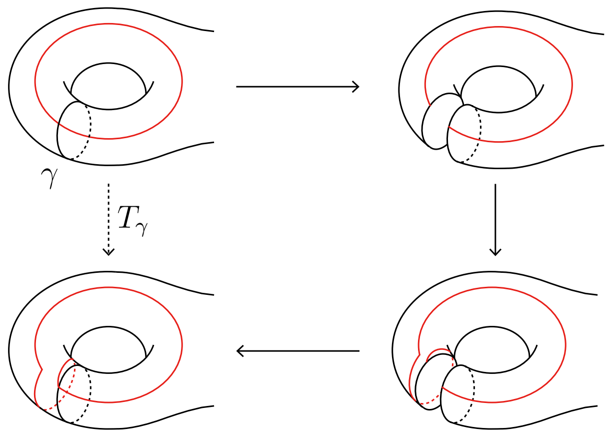

Figure 1.1: Dehn twist

representation is an infinite-index subgroup of the Torelli subgroup and it contains Dehn twists of all separating simple closed curves. For some pseudo-Anosov mapping class, we can find a train track invariant from the mapping class.

1.1 Main theorems

Part I Let T_γ denote the right-handed Dehn twist along a simple closed curve γ of a surface; see Figure 1.1. Let $i(\alpha, \beta)$ be the (geometric) intersection number of two simple closed curves α and β. The following theorem suggests several conditions to construct a right-angled Artin subgroup of a mapping class group.

Theorem 1 (Corollary 4.3.5, Corollary 4.3.4, Theorem 4.3.3)**.** *Let Σ be an orientable finite type surface and let $\gamma_1, \ldots, \gamma_k$ be finitely many simple closed curves on Σ. Then $T^n_{\gamma_1}, \ldots, T^n_{\gamma_k}$ generates a right-angled Artin subgroup of* $\mathrm{Mod}(\Sigma)$ *if one of the following holds.*

1. *$i(\gamma_i, \gamma_j) \leq 1$ for all $i, j \in \{1, \ldots, k\}$, and n is at least 7.*

2. *There exists $N \geq 2$ such that $i(\gamma_i, \gamma_j) \in \{0, N\}$ for every $i, j \in \{1, \ldots, k\}$, and n is at least 6.*

the Euclidean plane. However, there are large gap between two theories until now. One of my hope is a combination of these theories.

3. *$M := \max\{i(\gamma_i, \gamma_j) \mid i, j = 1, \dots, k\}$ is at least 2, and n is at least $M^2 + M + 3$.*

As an important step of the proof, we discover an isometry on the dual tree of a simple closed curve which imitates a lift of the Dehn twist. (Proposition 4.1.1) The isometry is a product of an elliptic isometry f and an element of the fundamental group. Furthermore, for each vertex v on the dual tree, there exists an element g_v of $\pi_1(\Sigma)$ such that $f(v) = g_v v$. (Proposition 4.1.2)

Meanwhile, for every geodesic of the hyperbolic plane, its projective image on the dual tree is either a vertex, a midpoint of an edge or a geodesic of the tree. (Proposition 3.1.3) The translation length of an element of $\pi_1(\Sigma)$ on the dual tree is related to the intersection number. (Lemma 3.2.2) Roughly speaking, the dual tree of a simple closed curve contains topological information about the curve.

Under this setting, a variation of the ping-pong lemma (for right-angled Artin group) can be applied for lifts of Dehn twists. If two simple closed geodesics α and β intersect each other transversally and a geodesic $\tilde{\gamma}$ of the hyperbolic plane is close to a lift of α, then a sufficiently large power of (a lift of) the Dehn twist along β changes $\tilde{\gamma}$ to a geodesic close to a lift of β. It can be verified on dual trees and we can compute how close they are by counting edges in detail.

Part II The main theme of Part II is a quasi-isometric action of the automorphism group of a surface group on the dual cube complex of simple closed curves. The dual cube complex of simple closed curves is a generalization of a dual tree, which is simply connected and satisfies a median property. Note that the quasi-isometry group of a metric space is the quotient of the set of quasi-isometries by the supremum norm, and a quasi-isometric action of a group on a metric space is a homomorphism from the group the quasi-isometry group of the metric space.

For every finite set of simple closed curves on a surface Σ, the dual cube complex of these curves is finite-dimensional. (Lemma 7.1.1) And the isometric action of $\pi_1(\Sigma)$ on the dual cube complex is cocompact and faithful. (Proposition 7.1.6 and Lemma 7.1.9) Note that $\pi_1(\Sigma)$ is isomorphic to the inner automorphism group of the automorphism group of $\pi_1(\Sigma)$. For each dual cube complex of curves, there is a subgroup of $\mathrm{Aut}(\pi_1(\Sigma))$ which acts faithfully on the cube complex by quasi-isometries. (Lemma 8.3.8)

Theorem 2. *For every closed orientable surface Σ of genus ≥ 2, there exist finitely many simple closed curves such that the automorphism group of $\pi_1(\Sigma)$, denoted by $\mathrm{Aut}(\pi_1(\Sigma))$, acts faithfully on the dual cube complex $\tilde{X}$ of these curves by quasi-isometries and the following diagram is commutative.*

$$\begin{tikzcd}
\pi_1(\Sigma) \arrow[r, hook] \arrow[d, hook] & \mathrm{Isom}(\tilde{X}) \arrow[d] \\
\mathrm{Aut}(\pi_1(\Sigma)) \arrow[r, hook] & \mathcal{QI}(\tilde{X})
\end{tikzcd}$$

A lift f of a Dehn twist permutes vertices of the dual cube complex (by sliding permutation). Similar to Proposition 4.1.2 in Part I, for every vertex v of the dual cube complex, there is an element g_v of $\pi_1(\Sigma)$ such that $f(v) = g_v v$. This property helps us show that f is quasi-isometric on the dual cube complex. Bridges are main ingredients of the proof of Theorem 2. The definition of bridge in this thesis is different from one of other papers such as Behrstock–Charney [1] and Chatterji–Fernos–Iozzi [12]. For two disjoint hyperplanes of a CAT(0) cube complex, if a geodesic intersects these hyperplanes, then this geodesic must cross the bridge. As every bridge is bounded, its diameter is used to compute the quasi-isometric constant for lifts of Dehn twists.

1.2 Future research

The contents of this thesis can be generalized to the study of affine transformations of CAT(0) cube complexes. Automorphisms of surface groups, automorphisms of free groups and all matrices in $\mathrm{SL}(n, \mathbb{Z})$ are affine transformations of some CAT(0) cube complexes.

A *cubical affine transformation* of a CAT(0) cube complex is a vertex permutation sending geodesics to geodesics (with respect to both the L^1-metric and the L^2-metric of the cube complex). The cubical affine group of a CAT(0) cube complex is the group of cubical affine transformations. The following is a recent work for a cubical affine group.

Pseudo-Anosov laminations There are several well-known ways to construct pseudo-Anosov mapping classes by Dehn twists. Penner [31] gave a way to make pseudo-Anosov mapping classes using two pants decompositions

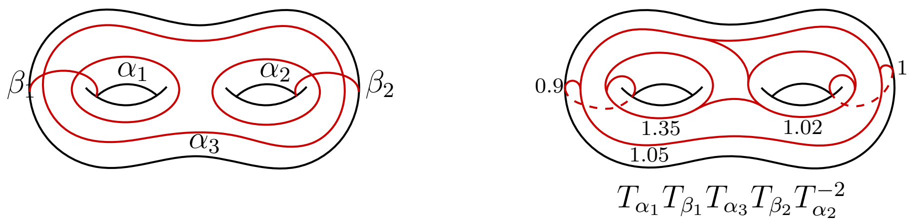

Figure 1.2

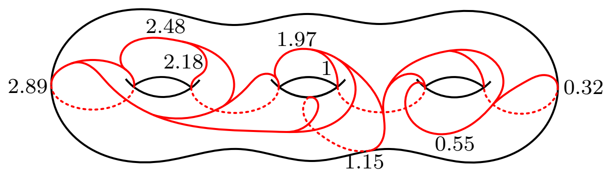

Figure 1.3

which are transverse to each other. In his proof, he constructed transverse pseudo-Anosov train tracks preserved by some words of Dehn twists and found a stretch factor by the Perron–Frobenius theorem.

We try to generalize Penner's algorithm to construct a train track for every word of Dehn twists. Although Shin–Strenner [39] showed that many pseudo-Anosov cannot be constructed by Penner's method, we hope that our tool can construct all pseudo-Anosov mapping classes.

This method is different from the algorithm of Bestvina–Handel [4], which can construct train tracks of all pseudo-Anosov mapping classes. We expect that there are subgroups H of $\mathrm{SL}(n,\mathbb{Z})$ and surjective homomorphisms $\varphi : H \to \mathrm{Mod}(\Sigma)$ such that an eigenvalue of $A \in H$ is the stretch factor of $\varphi(A)$ if $\varphi(A)$ is a pseudo-Anosov mapping class.

As an exercise, we calculated the stretch factor of a pseudo-Anosov mapping class. Precisely, the word $T_{\alpha_1}T_{\beta_1}T_{\alpha_3}T_{\beta_2}T_{\alpha_2}^{-2}$ in Figure 1.2 is pseudo-Anosov, (which can be proved by the Casson's homological criterion [11]). Then we calculated that the stretch factor of this word is exactly

$$\frac{1}{4}\left(3+\sqrt{13}+\sqrt{6(1+\sqrt{13})}\right) \approx 2.96557.$$

And the second picture of Figure 1.2 is a measured train tracks invariant from the word. Figure 1.3 is a measured train track of a minimal dilatation pseudo-Anosov mapping class on an orientable closed surface of genus 3: Lanneau–Thiffeault [26, Proposition 3.1] proved that this mapping class is a minimal dilatation pseudo-Anosov mapping class on a surface of genus 3.

1.3 Guide to the reader

Chapter 2 consists of basic facts of hyperbolic surfaces. We are going to analyze surfaces by hyperbolic geometry. This chapter discuss the definition of hyperbolic plane and mapping class group. Chapter 3 provides properties of tree actions of fundamental groups of surfaces, (especially, on dual trees of simple closed curves). The main theme of this chapter is the relationship between intersection numbers of simple closed curves and translation lengths of elements of surface groups; see Proposition 3.2.5. In Chapter 4, we prove the main theorem of Part I. We introduce the action of lifts of Dehn twists on the dual tree and answer how to solve the equation $fv = gv$; see Proposition 4.1.2. In Section 4.2, we define a set used for the ping-pong argument in the proof.

Chapter 5 is a preliminary chapter of CAT(0) cube complexes. In Chapter 6, the pocset structure of CAT(0) cube complexes and collapsings are dealt. We recover basic facts about convex subcomplexes in Section 6.2. In Section 6.3, we extend the definition of a bridge and give a criterion of horzontal hyperplanes in Proposition 6.3.13. Chapter 7 explains the actions of fundamental groups of surfaces on CAT(0) cube complexes. We introduce sliding permutation in Chapter 8. In Section 8.2, we construct sliding permutation on the 0-skeleton of CAT(0) cube complexes and develop the computation of sliding permutations on $\pi_1(\Sigma)$ in Proposition 8.2.5. And we prove that the sliding permutations of two disjoint simple closed geodesics commute with each other modulo $\pi_1(\Sigma)$ in Proposition 8.2.11. In Section 8.3, we show that $\pi_1(\Sigma)$ is embedded in the quasi-isometry groups of CAT(0) cube complexes in Lemma 8.3.6 and combined with Proposition 8.2.5.(3) and Theorem 8.3.1, we obtained the fact that every sliding permutation is a quasi-isometry in the normalizer of $\pi_1(\Sigma)$.

For some propositions, we place their proofs in the appendix. After the last section, we collect frequently used notations and symbols.

Part I

Dehn twists and dual trees

Chapter 2

Preliminaries for hyperbolic surfaces

2.1 Surfaces

Definition 2.1.1. A *closed surface* is a complete connected compact Hausdorff 2-dimensional (real) manifold without boundary. A *punctured surface* is the complement of finitely many points of a closed surface. We call these points *punctures* of the surface. A *finite type surface* is either a closed surface or a punctured surface.

For example, a sphere S^2, a torus T^2 and a Klein bottle are closed surfaces. By [32, Theorem 9.1.1], every closed surface is homeomorphic to either (i) a sphere, (ii) the connected sum of finitely many tori or (iii) the connected sum of finitely many Klein bottles. In these caeses, the surfaces of (i) and (ii) are *orientable* and the others are *unorientable*.

Definition 2.1.2. The *genus* of a closed orientable surface is the number of tori such that their connect sum is homeomorphic to the surface. If $\Sigma_{g,n}$ is a surface of genus g with n punctures, then the *complexity* of $\Sigma_{g,n}$ is $3g-3+n$.

Without mentioned, we write Σ as an orientable surface of finite type with positive complextiy.

2.2 Hyperbolic plane

Definition 2.2.1. The (Poincaré model[1] of the) *hyperbolic plane*, denoted by $\mathbb{H}^2$, is the open unit disk $\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < 1\}$ with the Riemannian metric

$$ds^2 := \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

By the above definition, for two points $p, q \in \mathbb{H}^2$, the hyperbolic distance $d_{\mathbb{H}^2}$ between p and q is

$$d_{\mathbb{H}^2}(p,q) = 2 \operatorname{arcsinh} \frac{|p-q|^2}{(1-|p|^2)(1-|q|^2)}.$$

The boundary of the disk represents the ideal boundary $\partial_\infty \mathbb{H}^2$ of the hyperbolic plane. Every geodesic of the Poincaré disk model is a circular line on the disk intersecting the ideal boundary orthogonally.

By the Brouwer fixed-point theorem, every isometry of the hyperbolic plane fixes some point on $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$.

Proposition 2.2.2 (Classification of isometries of the hyperbolic plane)**.** *Every non-identity isometry f of the hyperbolic plane is one of the following.*

1. *f is* elliptic*, that is, f fixes a point on $\mathbb{H}^2$.*
2. *f is* parabolic*, that is, f does not fix a point on $\mathbb{H}^2$ and it stabilizes a unique point on $\partial_\infty \mathbb{H}^2$.*
3. *f is* hyperbolic*, that is, f does not fix a point on $\mathbb{H}^2$ and it stabilizes two points on $\partial_\infty \mathbb{H}^2$.*

2.3 Hyperbolic structure

Lemma 2.3.1 (Exercise 9.8.5 in [32])**.** *For an orientable finite type surface Σ with positive complexity, there is a discrete subgroup G of the orientation-preserving isometry group of $\mathbb{H}^2$ such that*

[1] There are equivalent definitions of a hyperbolic space, which are isometric to each other: the hyperboloid modal, the Klein disk model and the upper half-plane model. For more detail, see [7, Chapter I.6] and [32].

1. *the quotient space* $\mathbb{H}^2/G$ *has finite area and*
2. *there is a homeomorphism between* $\mathbb{H}^2/G$ *and* Σ *corresponding each puncture of* Σ *to a cusp of* $\mathbb{H}^2/G$.

Remark 2.3.2. By the uniqueness of a universal cover, the group G in Lemma 2.3.1 is isomorphic to $\pi_1(\Sigma)$, the fundamental group of Σ.

Definition 2.3.3. For an orientable finite-type surface Σ with positive complexity, we call a covering map $\xi : \mathbb{H}^2 \to \Sigma$ a *hyperbolic structure* of Σ if the deck tranformation group of ξ is a discrete subgroup of the orientation-preserving isometry group of $\mathbb{H}^2$.

Remark 2.3.4. A thrice-punctured sphere $\Sigma_{0,3}$, whose complexity is zero, admits a unique hyperbolic structure; see [32, Theorem 9.8.8]. But every simple closed curve on $\Sigma_{0,3}$ is homotopic to a puncture so that $\Sigma_{0,3}$ is not under consideration.

2.4 Simple closed geodesics

Let Σ be an orientable finite type surface with positive complexity. And we give a hyperbolic structure on Σ by Lemma 2.3.1. In this subsection, we will look into a 1-dimensional submanifold of Σ, which is not homotopic to a point.

2.4.1 Simple closed geodesics and minimal position

Definition 2.4.1. A *simple closed curve* γ on Σ is the image of an embedding $S^1 \hookrightarrow \Sigma$. A simple closed curve is said to be *essential* if it is not homotopic to a point or a puncture of Σ.

Definition 2.4.2. For two simple closed curves γ and η, the *(geometric) intersection number* of γ and η, denoted by $i(\gamma, \eta)$, is

$$\min\{|\gamma' \cap \eta'| \mid \gamma' \text{ and } \eta' \text{ are homotopic to } \gamma \text{ and } \eta\text{, respectively}\}.$$

Then γ and η are *in minimal position* if $|\gamma \cap \eta| = i(\gamma, \eta)$.

For example, if γ_1 and γ_2 are parallel, i.e., bound some annulus in Σ, then the intersection number of γ_1 and γ_2 is zero.

We say that two simple closed curves γ_1 and γ_2 form a *bigon* if there exist subarcs $\alpha_1 \subset \gamma_1$ and $\alpha_2 \subset \gamma_2$ such that $\alpha_1 \cup \alpha_2$ is a simple closed curve which is homotopic to a point.

Lemma 2.4.3 (Bigon criterion, Proposition 1.7 in [17])**.** *Two simple closed curves γ_1 and γ_2 do not form any bigon if and only if they are in minimal position.*

Meanwhile, a *simple closed geodesic* in Σ is a simple closed curve which is the image of a geodesic in $\mathbb{H}^2$.

Lemma 2.4.4 (Theorem 9.6.5 in [32])**.** *For every essential simple closed curve γ of Σ, there is a unique simple closed geodesic which is homotopic to γ.*

Lemma 2.4.5 (Corollary 1.9 in [17])**.** *If $\gamma_1, \dots, \gamma_k$ are distinct simple closed geodesics, then they are in minimal position.*

Combining Lemma 2.4.4 with Lemma 2.4.5, we deduce that for finitely many simple closed curves $\gamma_1, \dots, \gamma_k$, which are not homotopic to each other, there are simple closed curves $\gamma'_1, \dots, \gamma'_k$ in minimal position such that γ_i is homotopic to γ'_i for each i.

2.4.2 Simple closed geodesics and halfspaces

Definition 2.4.6. Given a geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$, the closure of a connected component of $\mathbb{H}^2 \setminus \tilde{\gamma}$ is called a *(closed) halfspace* of $\tilde{\gamma}$. For a halfspace H of $\tilde{\gamma}$, we write H^* as the other haflspace of $\tilde{\gamma}$.

Definition 2.4.7. For every geodesic γ of Σ, a *lift* $\tilde{\gamma}$ of γ is a geodesic on $\mathbb{H}^2$ such that $\xi(\tilde{\gamma}) = \gamma$ where ξ is a covering map.

Lemma 2.4.8. *Let γ be a simple closed geodesic in Σ. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be distinct lifts of γ. If H_1 and H_2 are halfspaces of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, respectively, then one of the inclusions $\mathsf{H}_1 \subsetneq \mathsf{H}_2$, $\mathsf{H}_1 \supsetneq \mathsf{H}_2$, $\mathsf{H}_1 \subsetneq \mathsf{H}_2^*$ and $\mathsf{H}_1 \supsetneq \mathsf{H}_2^*$ is satisfied.*

Proof. Because γ does not have any self-intersection, $\tilde{\gamma}_1$ is disjoint from $\tilde{\gamma}_2$. So either $\tilde{\gamma}_1 \subset \mathsf{H}_2$ or $\tilde{\gamma}_1 \subset \mathsf{H}^*$. If H_2 contains $\tilde{\gamma}_1$ as a subset, then either $\mathsf{H}_1 \subsetneq \mathsf{H}_2$ or $\mathsf{H}_1 \supsetneq \mathsf{H}_2^*$. If H_2^* contains $\tilde{\gamma}_1$, then either $\mathsf{H}_1 \subsetneq \mathsf{H}_2^*$ or $\mathsf{H}_1 \supsetneq \mathsf{H}_2$. $\square$

2.5 Collar lemma

Let Σ be an orientable hyperbolic surface of finite area. For a simple closed geodesic γ in Σ, let $l(\gamma)$ denote the length of γ. Note that γ is compact so that there is a positive number $r > 0$ such that the r-neighborhood of γ is homeomorphic to an annulus. The collar lemma below says that $\{r > 0 \,|\, \text{the } r\text{-neighborhood of } \gamma \text{ is an annulus}\}$ is bounded above by some number depending only on $l(\gamma)$.

Lemma 2.5.1 (Collar lemma, (A) and (B) in [9]). *Let γ and γ' be simple closed geodesics of a hyperbolic surface Σ.*

1. *For every $R < \cosh^{-1}\left(\coth\frac{l(\gamma)}{2}\right)$, the closed R-neighborhood of γ is homeomorphic to an annulus.*

2. *The distance between γ and γ' is larger than*

$$\cosh^{-1}\left(\coth\frac{l(\gamma)}{2}\right) + \cosh^{-1}\left(\coth\frac{l(\gamma')}{2}\right).$$

Definition 2.5.2. For each simple closed geodesic γ in a hyperbolic surface Σ, the constant $C(\gamma) := \cosh^{-1}(\coth(l(\gamma)/2))$ is said to be the *collar length* of γ.

2.6 Mapping class groups and Dehn twists

Definition 2.6.1. Let Σ be an orientable finite type surface of positive complexity. The *mapping class group* of Σ, denoted by $\mathrm{Mod}(\Sigma)$, is the group of homotopy classes of orientation-preserving (self-)homeomorphisms of Σ. That is,

$$\mathrm{Mod}(\Sigma) := \mathrm{Homeo}^+(\Sigma)/\sim$$

where $f_1 \sim f_2$ means that f_1 is homotopic to f_2.

Consider the annulus $(\mathbb{R}/\mathbb{Z}) \times [0,1]$. And we define a homeomorphism $T : (\mathbb{R}/\mathbb{Z}) \times [0,1] \to (\mathbb{R}/\mathbb{Z}) \times [0,1]$ by $T([s],t) = ([s+t],t)$ for all $([s],t) \in (\mathbb{R}/\mathbb{Z}) \times [0,1]$.

Definition 2.6.2. Let γ be an essential simple closed curve. Let $\mathcal{N}(\gamma)$ be a regular neighborhood of γ and $\phi : (\mathbb{R}/\mathbb{Z}) \times [0,1] \to \mathcal{N}(\gamma)$ an orientation-preserving homeomorphsm. Then we say that the mapping class of

$$T_\gamma(p) := \begin{cases} (\phi \circ T \circ \phi^{-1})(p) & \text{if } p \in \mathcal{N}(\gamma), \\ p & \text{if } p \in \Sigma \setminus \mathcal{N}(\gamma) \end{cases}$$

is the (right-handed) *Dehn twist* along γ.

Lemma 2.6.3 (Proposition 3.1 and Section 3.3. in [17])**.** *Let γ and η be essential simple closed curves in Σ. For Dehn twists T_γ and T_η along γ and η, respectively, the following hold.*

1. *The n-th power of T_γ is equal to the identity in* $\mathrm{Mod}(\Sigma)$ *if and only if $n = 0$.*
2. *T_γ is equal to T_η if and only if γ is homotopic to η.*
3. *For all $n \in \mathbb{Z}$ and $F \in \mathrm{Mod}(\Sigma)$, we have $T^n_{F(\gamma)} = FT^n_\gamma F^{-1}$.*
4. *Let $F \in \mathrm{Mod}(\Sigma)$ be given. The commutator $[F, T_\gamma] = FT_\gamma F^{-1}T_\gamma^{-1}$ is equal to the identity if and only if $F(\gamma) = \gamma$. In particular, $[T_\gamma, T_\eta] = 1$ if and only if $i(\gamma, \eta) = 0$.*

2.6.1 Outer automorphisms of fundamental groups of surfaces

There is a faithful action of $\mathrm{Mod}(\Sigma)$ on the set of conjugacy classes of $\pi_1(\Sigma)$, which gives the natural injection $\mathrm{Mod}(\Sigma) \hookrightarrow \mathrm{Out}(\pi_1(\Sigma))$. Let us observe the detail. Let F be a self-homeomorphism of Σ and let $\tilde{F} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of F. For every $g \in \pi_1(\Sigma)$ and a point $p \in \mathbb{H}^2$, we have

$$\xi(\tilde{F}g\tilde{F}^{-1}p) = F\xi(g\tilde{F}^{-1}p) = F\xi(\tilde{F}^{-1}p) = FF^{-1}\xi(p) = \xi(p).$$

It implies that $\tilde{F}g\tilde{F}^{-1}$ is an element of $\pi_1(\Sigma)$, i.e., $\tilde{F}$ acts on $\pi_1(\Sigma)$ as an isomorphism. If we choose another lift of F, it can be written by $h\tilde{F}$ for some $h \in \pi_1(\Sigma)$ and $h\tilde{F}g(h\tilde{F})^{-1} = h(\tilde{F}g\tilde{F}^{-1})h^{-1}$ for all $g \in \pi_1(\Sigma)$. So the set $I(F) := \{g \mapsto \tilde{F}g\tilde{F}^{-1} \mid \tilde{F} \text{ is a lift of } F\}$ is an outer automorphism of $\pi_1(\Sigma)$, that is, $I(F) \in \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma))$. Then the map $I : \mathrm{Homeo}^+(\Sigma) \to$

$\mathrm{Out}(\pi_1(\Sigma))$ is a homomorphism with the kernel $\mathrm{Homeo}_0(\Sigma)$. Therefore, by the isomorphism theorem, $\mathrm{Mod}(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma))$ is an injective homomorphism.

Lemma 2.6.4 (Section 8.1 in [17])**.** *There is a natural injective homomorphism* $\mathrm{Mod}(\Sigma) \hookrightarrow \mathrm{Out}(\pi_1(\Sigma))$*.*

By the Dehn-Nielsen-Baer theorem in [17, Theorem 8.1], if Σ is a closed surface, then the image of the map $\mathrm{Mod}(\Sigma) \hookrightarrow \mathrm{Out}(\pi_1(\Sigma))$ has index 2 in $\mathrm{Out}(\pi_1(\Sigma))$.

Chapter 3

Tree actions of fundamental groups of surfaces

3.1 Dual trees of simple closed geodesics

Let Σ be an orientable hyperbolic surface of finite area. And let $\xi : \mathbb{H}^2 \to \Sigma$ be a covering map which is a local isometry.

Definition 3.1.1. Let γ be a simple closed geodesic on Σ. The *dual tree* of γ is a tree $\mathcal{Y}_\gamma$ with a metric d_γ and an action σ_γ of $\pi_1(\Sigma)$ defined by the following.

- Let $\mathcal{V}_\gamma$ be the collection of connected components of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$. And let $\mathcal{E}_\gamma$ be the collection of pairs of components in $\mathcal{V}_\gamma$ satisfying that

$$\{V_1, V_2\} \in \mathcal{E}_\gamma \Longleftrightarrow \bar{V}_1 \cap \bar{V}_2 \neq \emptyset$$

  where $\bar{V}_1$ and $\bar{V}_2$ are the closures of V_1 and V_2, respectively. Then $\mathcal{Y}_\gamma$ is the realization of the 1-dimensional abstract simplicial complex $(\mathcal{V}_\gamma, \mathcal{E}_\gamma)$.[1]

- d_γ is the metric on $\mathcal{Y}_\gamma$ such that every edge is of length 1.

- $\sigma_\gamma : \pi_1(\Sigma) \to \mathrm{Isom}(\mathcal{Y}_\gamma, d_\gamma)$ is the isometric action of $\pi_1(\Sigma)$ such that $\sigma_\gamma(g)V = gV$ for all $g \in \pi_1(\Sigma)$ and $V \in \mathcal{V}_\gamma$.

[1] Morgan–Shalen [30] proved that the dual tree $\mathcal{Y}_\gamma$ of a simple closed geodesic γ is indeed a tree.

Remark 3.1.2. Let γ be a simple closed geodesic of Σ, and let $\mathcal{V}_\gamma$ be the collection of connected components of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$. Then there is an inclusion $\mathcal{Y}_\gamma \hookrightarrow \mathbb{H}^2$ such that for every $V \in \mathcal{V}_\gamma$, the vertex of $\mathcal{Y}_\gamma$ corresponding to V lies on V and that each edge of $\mathcal{Y}_\gamma$ intersects exactly one lift of γ transversally. Passing to homotopy, we assume that for each edge, the midpoint of the edge is the intersection between the edge and a lift of γ.

An inclusion $\mathcal{Y}_\gamma \hookrightarrow \mathbb{H}^2$ in the above remark does not preserve actions of $\pi_1(\Sigma)$ between them because $\pi_1(\Sigma)$ acts on $\mathbb{H}^2$ freely but not on $\mathcal{Y}_\gamma$. Because both $\mathcal{Y}_\gamma$ and $\mathbb{H}^2$ are contractible, there is a deformation retract $\mathbb{H}^2 \to \mathcal{Y}_\gamma$ with respect to the inclusion; see the below proposition.

Proposition 3.1.3 (Proposition A.1.1)**.** *Let γ be a simple closed geodesic of Σ, and let $\mathcal{Y}_\gamma$ be the dual tree of γ. If ϵ is a positive number such that the ϵ-neighborhood of γ is homeomorphic to a cylinder, then there is a $\pi_1(\Sigma)$-equivariant surjective continuous map $\Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ such that the following hold.*

1. *For every edge e of $\mathcal{Y}_\gamma$, if u is a point in the interior of e, then the inverse image $\Phi^{-1}_{\gamma,\epsilon}(u)$ is a 1-dimensional subspace contained in the ϵ-neighborhood of some lift $\tilde{\gamma}$ of γ. And, in this case, $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$ is the midpoint of e.*

2. *For another simple closed geodesic α, there is $0 < \epsilon' \leq \epsilon$ such that for every lift $\tilde{\alpha}$ of α,*

   (a) $\Phi_{\gamma,\epsilon'}(\tilde{\alpha})$ is the midpoint of an edge if $\alpha = \gamma$,

   (b) $\Phi_{\gamma,\epsilon'}(\tilde{\alpha})$ is a vertex if α is disjoint from γ, or

   (c) $\Phi_{\gamma,\epsilon'}(\tilde{\alpha})$ is a geodesic if α intersects γ transversally.

To prove Proposition 3.1.3 precisely, we will use the pocset structure; see Definition 5.5.1 for pocset structure; see Appendix A.1 for the proof of Proposition 3.1.3. You can also find a more conceptual proof of Proposition 3.1.3 in the author's arXiv paper [37, Proposition 2.3].

3.2 The fundamental groups of surfaces acting on dual trees

Definition 3.2.1. Let f be an isometry on a tree $\mathcal{Y}$ with a metric d. The *translation length* of f, denoted by $\operatorname{tr} f$, is the number

$$\inf_{v \in \mathcal{Y}} d(v, fv).$$

For every simple closed geodesic γ and $g \in \pi_1(\Sigma)$, we write the translation length of $\sigma_\gamma(g)$ as $\operatorname{tr}_\gamma g$ instead of $\operatorname{tr} \sigma_\gamma(g)$.

A *primitive element* of a group $\pi_1(\Sigma)$ is an element of $\pi_1(\Sigma)$ which cannot be written by nontrivial power of an element of $\pi_1(\Sigma)$. Then the translation length of a primitive element of $\pi_1(\Sigma)$ on the dual tree of γ is exactly the intersection number between its support and γ.

Lemma 3.2.2. *Let α and β be simple closed geodesics on Σ. If h is a primitive element of $\pi_1(\Sigma)$ whose axis on $\mathbb{H}^2$ is a lift of α, then the translation length of h on the dual tree of β is exactly the intersection number between α and β, that is,*

$$\operatorname{tr}_\beta h = i(\alpha, \beta).$$

Proof. Let $\tilde{\alpha}$ be the lift of α preserved by h. If $i(\alpha, \gamma) = 0$, then $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ is a point on $\mathcal{Y}_\gamma$ for some $\epsilon > 0$. Because h preserves $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$, we have $\operatorname{tr}_\gamma h = 0 = i(\alpha, \gamma)$.

Assume that α crosses γ. By Proposition 3.1.3, it is satisfied that h preserves the geodesic $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ for some $r > 0$. If p is a point on $\tilde{\alpha}$ such that $\Phi_{\gamma,\epsilon}(p)$ is a vertex, then $\operatorname{tr}_\gamma h$ is equal to $d_\gamma(\Phi_{\gamma,\epsilon}(p), \sigma_\gamma(h)\Phi_{\gamma,\epsilon}(p))$ by Bass-Serre theory. Since the geodesic segment joining p and hp crosses exactly $i(\alpha, \gamma)$ lifts of γ, we have $d_\gamma(\Phi_{\gamma,\epsilon}(p), \Phi_{\gamma,\epsilon}(hp)) = i(\alpha, \gamma)$. Therefore, $\operatorname{tr}_\gamma h = i(\alpha, \gamma)$. $\square$

A *geodesic triangle* on Σ or on $\mathbb{H}^2$ is a union of three geodesic segments $\alpha_1 \cup \alpha_2 \cup \alpha_3$ such that α_i shares an endpoint with α_j for each $i, j \in \{1, 2, 3\}$.

Lemma 3.2.3. *Let α, β and γ be simple closed geodesics on Σ intersecting each other transversally. If x, y and z are points in $\alpha \cap \beta$, $\beta \cap \gamma$ and $\gamma \cap \alpha$, respectively, then the number of distinct contractible geodesic triangles whose vertices are x, y and z is at most* 1.

Proof. For contradiction, assume that there are two distinct contractible geodesic triangles Δ_1 and Δ_2 such that each of them has vertices x, y and z. Let D_1 and D_2 be disks whose boundaries are Δ_1 and Δ_2, respectively.

If $\Delta_1 \cap \Delta_2 = \{x, y, z\}$, then $D_1 \cup D_2$ is homotopic to a pair of pants (a sphere with three boundary components). It implies that α, β and γ are pairwise disjoint, which is a contradiction. So Δ_1 shares a side with Δ_2.

If $\Delta_1 \cap \Delta_2 = \Delta_1$, then $\Delta_1 \cup \Delta_2$ contains a sphere which is not homotopic to a point It is a contradiction because the second homology of Σ is trivial.

If $\Delta_1 \cap \Delta_2$ is the union of exactly two sides of Δ_1, then $D_1 \cup D_2$ is a disk and its boundary is one of α, β and γ. So one of α, β and γ is null-homotopic, which is a contradiction.

If $\Delta_1 \cap \Delta_2$ is the disjoint union of a vertex and a side of Δ_1, then $D_1 \cup D_2$ is homotopic to a cylinder. So for some pair in $\{\alpha, \beta, \gamma\}$, they are homotopic to each other, which is a contradiction. As a result, there are not two distinct contractible geodesic triangles of vertices x, y and z. $\square$

Definition 3.2.4. Let γ be a simple closed geodesic on Σ. If L_1 and L_2 are geodesics on $\mathbb{H}^2$, then we write $\Delta_\gamma(L_1, L_2)$ as the number of lifts of γ intersecting both L_1 and L_2 transversally.

Proposition 3.2.5. *Let α, β and γ be simple closed geodesics on Σ. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β, respectively. Then the following hold.*

1. *Let $\Phi_{\gamma,\epsilon}$ be the map satisfying Proposition 3.1.3 for some sufficiently small $\epsilon > 0$. Then $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta})$ is equal to the length of $\Phi_{\gamma,\epsilon}(\tilde{\alpha}) \cap \Phi_{\gamma,\epsilon}(\tilde{\beta})$.*
2. *If $\tilde{\alpha}$ is disjoint from $\tilde{\beta}$, then $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta})$ is at most* $\min\{i(\alpha, \gamma), i(\beta, \gamma)\}$.
3. *If $\tilde{\alpha}$ intersects $\tilde{\beta}$ transversally, then $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta})$ is at most* $\mathrm{lcm}\{i(\alpha, \gamma), i(\beta, \gamma)\}$.

Proof. (1) It is enough to show that a lift $\tilde{\gamma}$ of γ intersects both $\tilde{\alpha}$ and $\tilde{\beta}$ transversally if and only if the edge e containing $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$ is contained in $\Phi_{\gamma,\epsilon}(\tilde{\alpha}) \cap \Phi_{\gamma,\epsilon}(\tilde{\beta})$.

If a lift $\tilde{\gamma}$ of γ intersects both $\tilde{\alpha}$ and $\tilde{\beta}$, then, because $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$ is a point, it is contained in $\Phi_{\gamma,\epsilon}(\tilde{\alpha}) \cap \Phi_{\gamma,\epsilon}(\tilde{\beta})$. Note that $\gamma \in \{\alpha, \beta\}$ since γ intersects α and β transversally. So $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ and $\Phi_{\gamma,\epsilon}(\tilde{\beta})$ are geodesic on $\mathcal{Y}_\gamma$ by Proposition 3.1.3. So the intersection $\Phi_{\gamma,\epsilon}(\tilde{\alpha}) \cap \Phi_{\gamma,\epsilon}(\tilde{\beta})$ contains the edge whose midpoint is $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$.

Conversely, suppose that there is a lift $\tilde{\gamma}$ of γ such that the edge containing $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$ is contained in $\Phi_{\gamma,\epsilon}(\tilde{\alpha}) \cap \Phi_{\gamma,\epsilon}(\tilde{\beta})$. Then α and β intersect γ

transversally by Proposition 3.1.3. It implies that $\tilde{\gamma} \notin \{\tilde{\alpha}, \tilde{\beta}\}$. Because $\Phi_{\gamma,\epsilon}(\tilde{\gamma})$ is contained in both $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ and $\Phi_{\gamma,\epsilon}(\tilde{\beta})$, we have $\tilde{\gamma}$ intersects both $\tilde{\alpha}$ and $\tilde{\beta}$ transversally.

(2) Let $\epsilon > 0$ be sufficiently small, and let $\Phi_\gamma = \Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 3.1.3. Suppose that either $i(\alpha, \gamma)$ or $i(\beta, \gamma)$ is zero. Then $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$ has at most one element. So $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) = 0$.

Assume that both $i(\alpha, \gamma)$ and $i(\beta, \gamma)$ are positive. Let h be a primitive element of $\pi_1(\Sigma)$ preserving $\tilde{\alpha}$. Because h is orientation-preserving, $\tilde{\alpha}$ does not separate $\tilde{\beta}$ from $h\tilde{\beta}$. Since h is an isometry, $h\tilde{\beta}$ does not separate $\tilde{\alpha}$ from $\tilde{\beta}$. Likewise, $\tilde{\beta}$ does not separate $\tilde{\alpha}$ from $h\tilde{\beta}$. By Remark 7.1.10, for every lift of γ, it is disjoint from some of $\tilde{\alpha}$, $\tilde{\beta}$ and $h\tilde{\beta}$.

Then $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \cap \Phi_\gamma(h\tilde{\beta})$ is a vertex or is empty. So the intersection between $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$ and $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(h\tilde{\beta}) = \sigma_\gamma(h)(\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}))$ does not contain an edge. Similarly, $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$ does not share an edge with $\sigma_\gamma(h^{-1})(\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}))$. It implies that $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$ is the line segment of length at most $\operatorname{tr}_\gamma h = i(\alpha, \gamma)$.

In other words, the number of lifts of γ intersecting $\tilde{\alpha}$ and $\tilde{\beta}$ simultaneously is at most $i(\alpha, \gamma)$. Changing the role between α and β, we have $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq i(\beta, \gamma)$. Therefore, $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq \min\{i(\alpha, \gamma), i(\beta, \gamma)\}$.

(3) Write $N := \operatorname{lcm}\{i(\alpha, \gamma), i(\beta, \gamma)\}$, $a := N/i(\alpha, \gamma)$ and $b := N/i(\beta, \gamma)$. For contradiction, assume that $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \geq N + 1$. Let h be a primitive element of $\pi_1(\Sigma)$ preserving $\tilde{\alpha}$. By Proposition 3.1.3 and the assumption, $\Phi_\gamma(\tilde{\alpha})$ and $\Phi_\gamma(\tilde{\beta})$ are geodesics on $\mathcal{Y}_\gamma$ and their intersection is a line segment of length at least $N + 1$. So there is an edge e on $\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$ such that $\sigma_\gamma(h^a)e \subset \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})$. In other words, there is a lift $\tilde{\gamma}$ of γ such that both $\tilde{\gamma}$ and $h^a\tilde{\gamma}$ crosses both $\tilde{\alpha}$ and $\tilde{\beta}$.

Because $d_\gamma(\Phi_\gamma(\tilde{\gamma}), \Phi_\gamma(h^a\tilde{\gamma}))$ is a multiple of $i(\beta, \gamma)$, there is a primitive element g of $\pi_1(\Sigma)$ preserving $\tilde{\beta}$ such that $g^b\tilde{\gamma} = h^a\tilde{\gamma}$. Let A and B be the geodesic triangles which are contained in $\tilde{\alpha} \cup \tilde{\beta} \cup \tilde{\gamma}$ and $\tilde{\alpha} \cup \tilde{\beta} \cup g^b\tilde{\gamma}$, respectively. If $\xi : \mathbb{H}^2 \to \Sigma$ is a covering map, $\xi(A)$ and $\xi(B)$ are distinct contractible geodesic triangles on Σ such that they share all vertices. It is a contradiction because of Lemma 3.2.3. Therefore, $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq N = \operatorname{lcm}\{i(\alpha, \gamma), i(\beta, \gamma)\}$. $\square$

Chapter 4

Lifts of Dehn twists

Let Σ be an orientable hyperbolic surface of finite type. And we fix a universal cover $\xi : \mathbb{H}^2 \to \Sigma$ which is a local isometry. It gives an inclusion $\pi_1(\Sigma) \to \mathbb{H}^2$ whose image is a discrete subgroup (precisely, a lattice) of $\mathrm{Isom}^+(\mathbb{H}^2)$. We identify $\pi_1(\Sigma)$ with the image.

If γ is a simple closed geodesic on Σ, we write $\mathcal{Y}_\gamma$ as the dual tree of γ in Definition 3.1.1 with the edge metric d_γ. To distinguish the action of $\pi_1(\Sigma)$ on $\mathcal{Y}_\gamma$ from the action on $\mathbb{H}^2$, we write σ_γ as the action of $\pi_1(\Sigma)$ on $\mathcal{Y}_\gamma$.

For every homeomorphism $F : \Sigma \to \Sigma$, a lift of F on $\mathbb{H}^2$ will be written as $\tilde{F}$. For example, in many case, we deal with a Dehn twist homeomorphism T_γ along γ and then $\tilde{T}_\gamma$ is a lift of T_γ. Note that the map $g \mapsto \tilde{F} g \tilde{F}^{-1}$ is an automorphism of $\pi_1(\Sigma)$.

In this chapter, we focus on lifts of Dehn twists and will show that for a lift $\tilde{T}_\gamma$ of the Dehn twist T_γ, there is an isometry f of the dual tree $\mathcal{Y}_\gamma$ such that $\sigma_\gamma(\tilde{T}_\gamma g \tilde{T}_\gamma^{-1}) = f \sigma_\gamma(g) f^{-1}$ for all $g \in \pi_1(\Sigma)$; see Proposition 4.1.2. Then we can extend the action of $\pi_1(\Sigma)$ on $\mathcal{Y}_\gamma$ to the action of $\langle \tilde{T}_\gamma, \pi_1(\Sigma) \rangle$.

For every finite set of simple closed geodesics is given, we show that there is a rational polynomial with respect to intersection numbers which gives a lower bound of exponents of powers of Dehn twists generating right-angled Artin groups.

4.1 Automorphisms of Dehn twist classes and elliptic isometries on dual trees

Proposition 4.1.1. *For a simple closed geodesic γ on Σ and a lift $\tilde{T}_\gamma$ of a Dehn twist T_γ along γ, there is a unique isometry f on the dual tree $\mathcal{Y}_\gamma$ of γ such that $\sigma_\gamma(\tilde{T}_\gamma g \tilde{T}_\gamma^{-1})(v) = f\sigma_\gamma(g)f^{-1}(v)$ for all $g \in \pi_1(\Sigma)$ and $v \in \mathcal{Y}_\gamma$.*

The proof of Proposition 4.1.1 is on Appendix A.2. If you want a proof without pocset structure, we recommend the proof of [37, Proposition 3.3]. From now on, we write $\sigma_\gamma(\tilde{T}_\gamma)$ instead of the isometry f. For a vertex v, we write $\mathrm{st}(v)$ as the closed 1-neighborhood of v.

Proposition 4.1.2. *Let γ be a simple closed geodesic on Σ, and let v be a vertex of the dual tree $\mathcal{Y}_\gamma$. Then there is a lift $\tilde{T}_\gamma$ of a Dehn twist T_γ such that*

$$\mathrm{st}(v) = \{u \in \mathcal{Y}_\gamma \mid \sigma_\gamma(\tilde{T}_\gamma)u = u\}.$$

Furthermore, if w is a vertex of $\mathcal{Y}_\gamma$ distinct from v and $\vec{L} : [0, n] \to \mathcal{Y}_\gamma$ is the unit-speed geodesic path from v to w, then there are primitive elements $h_1, \ldots, h_n \in \pi_1(\Sigma)$ such that h_i fixes the midpoint $\vec{L}(i - 1/2)$ for each $i = 1, \ldots, n$ and such that $\sigma_\gamma(\tilde{T}_\gamma^m)w = \sigma_\gamma(h_1^m \ldots h_n^m)w$ for every integer m.

Proof. Let $\epsilon > 0$ be small enough that the closed ϵ-neighborhood of γ is homeomorphic to an annulus. Let $\Phi_\gamma := \Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 3.1.3. Passing to homotopy, suppose that T_γ is a Dehn twist along γ such that the support of $T_\gamma(\gamma)$ is the ϵ-neighborhood of γ. Let p be a point on $\mathbb{H}^2$ such that $\Phi_\gamma(p) = v$.

Because the projective image of p on Σ is not contained in the open support of T_γ, there is a lift $\tilde{T}_\gamma$ of T_γ which fixes p. For every edge e incident to v, if $\tilde{\gamma}$ is the lift of γ satisfying that $\Phi_\gamma(\tilde{\gamma})$ is the midpoint of e, then $\tilde{T}_\gamma(\tilde{\gamma}) = \tilde{\gamma}$ because some connected component of the boundary of the ϵ-neighborhood of $\tilde{\gamma}$ is fixed pointwise by $\tilde{T}_\gamma$. So $\sigma_\gamma(\tilde{T}_\gamma)$ fixes e. Therefore, $\tilde{T}_\gamma$ fixes $\mathrm{st}(v)$ pointwise.

To prove the second statement of Proposition 4.1.2, we use an induction on the distance n between v and w. If v and w are joined by an edge e and a primitive element h of $\pi_1(\Sigma)$ fixes the edge e, then we have $\sigma_\gamma(\tilde{T}_\gamma^m)w = w = \sigma_\gamma(h^m)w$ for every integer m by the above.

Assume that $n = d_\gamma(v, w) \geq 2$. Let an integer m be given. Let $\tilde{T}'_\gamma$ be the lift of T_γ fixing $\mathrm{st}(\vec{L}(1))$ pointwise. By the induction hypothesis, there are primitive elements $h_2, \dots, h_n \in \pi_1(\Sigma)$ fixing midpoints $\vec{L}(3/2), \dots, \vec{L}(n-1/2)$, respectively, such that $\sigma_\gamma((\tilde{T}'_\gamma)^m)(w) = \sigma_\gamma(h_2^m \dots h_n^m)w$.

We claim that $(\tilde{T}'_\gamma)^m = h_1^{-m}\tilde{T}_\gamma^m$ for some primitive element h_1 fixing $\vec{L}(1/2)$. Let $\tilde{\gamma}$ be the lift of γ such that $\Phi_\gamma(\tilde{\gamma}) = \vec{L}(1/2)$. Then $\tilde{T}'_\gamma$ fixes some boundary component B of $\mathcal{N}_\epsilon(\tilde{\gamma})$ of the closed ϵ-neighborhood of $\tilde{\gamma}$ pointwise. By Definition A.2.1, there is a primitive element $h_1 \in \pi_1(\Sigma)$ preserving $\tilde{\gamma}$ such that $\tilde{T}_\gamma^m(p) = h_1^m p$ for every $p \in B$. Because $h_1^{-m}\tilde{T}_\gamma^m$ fixes p and is a lift of T_γ^m, we have $h_1^{-m}\tilde{T}_\gamma^m = (\tilde{T}'_\gamma)^m$ by the path-lifting property. Therefore, $\sigma_\gamma(\tilde{T}_\gamma^m)w = \sigma_\gamma(h_1^m h_2^m \dots h_n^m)w$. □

4.2 Dehn twist automorphisms and multicurves

Definition 4.2.1. A *multicurve* on Σ is a collection of pairwise disjoint simple closed geodesics on Σ. (The reason we call it a multicurve instead of *multigeodesic* is that it is a general notation.) A *lift* of a multicurve is a lift of a simple closed geodesic in the multicurve.

Definition 4.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be multicurves on Σ, and let α and β be simple closed geodesics in $\mathcal{A}$ and $\mathcal{B}$, respectively. For $n > 2$, a simple closed geodesic γ of Σ is said to be contained in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$ if there are some lift $\tilde{\gamma}$ of γ, some lift $\tilde{\alpha}$ of α and (at least) n lifts $\tilde{\beta}_1, \dots, \tilde{\beta}_n$ of $\mathcal{B}$ such that the following hold.

1. $\tilde{\alpha}$ intersects $\tilde{\gamma}$ transversally.
2. For every $i \in \{1, \dots, n\}$, the lift $\tilde{\beta}_i$ separates $\tilde{\beta}_{i-1}$ from $\tilde{\beta}_{i+1}$.
3. There is an index $i_0 \in \{2, \dots, n-1\}$ such that $\tilde{\beta}_{i_0}$ is a lift of β.
4. For all $i = 1, \dots, n$, the lift $\tilde{\beta}_i$ intersects both $\tilde{\alpha}$ and $\tilde{\gamma}$ transversally.
5. For a lift $\tilde{\alpha}'$ of $\mathcal{A}$, if there is $i \in \{1, \dots, n\}$ such that $\tilde{\alpha}'$ intersects $\tilde{\gamma}$ and $\tilde{\beta}_i$, then $\tilde{\alpha}' = \tilde{\alpha}$.

Lemma 4.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be multicurves, and let α and β be simple closed geodesics on $\mathcal{A}$ and $\beta \in \mathcal{B}$, respectively. For $n > 2$, the following hold.*

1. *If $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$ is nonempty, then $i(\alpha, \beta) > 0$.*

2. *If $2 < m \leq n$, then $\mathrm{PP}_m(\mathcal{A}, \alpha, \mathcal{B}, \beta)$ contains $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$ as a subset.*

3. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are multicurves satisfying that $\alpha \in \mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\mathrm{PP}_n(\mathcal{A}_1, \alpha, \mathcal{B}, \beta)$ contains $\mathrm{PP}_n(\mathcal{A}_2, \alpha, \mathcal{B}, \beta)$ as a subset.*

4. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are multicurves satisfying that $\beta \in \mathcal{B}_1 \subseteq \mathcal{B}_2$, then the set $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_1, \beta)$ is a subset of $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_2, \beta)$.*

Proof. (1) If some simple closed geodesic is contained in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$, then $\tilde{\alpha}$ intersects $\tilde{\beta}$ transversally for some lift $\tilde{\alpha}$ of α and lift $\tilde{\beta}$ of β by Definition 4.2.2(3) and (4). It implies that $i(\alpha, \beta) > 0$.

(2) Let γ be a simple closed geodesic in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$. Then there are a lift $\tilde{\gamma}$ of γ, a lift $\tilde{\alpha}$ of α and n lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\mathcal{B}$ satisfying (1)–(5) of Definition 4.2.2. Let $i_0 \in \{2, \ldots, n-1\}$ be given such that $\tilde{\beta}_{i_0}$ is a lift of β. Choose a subsequence $\tilde{\beta}_{j_1}, \ldots, \tilde{\beta}_{j_m}$ of length m from $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ such that $\tilde{\beta}_1$, $\tilde{\beta}_{i_0}$ and $\tilde{\beta}_n$ are contained in the subsequence. Then $\tilde{\gamma}$, $\tilde{\alpha}$ and the subsequence $\tilde{\beta}_{j_1}, \ldots, \tilde{\beta}_{j_m}$ satisfy (1)–(5) of Definition 4.2.2 of PP_m for γ. Therefore, $\gamma \in \mathrm{PP}_m(\mathcal{A}, \alpha, \mathcal{B}, \beta)$ for all $\gamma \in \mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$.

(3) Let a simple closed geodesic $\gamma \in \mathrm{PP}_n(\mathcal{A}_2, \alpha, \mathcal{B}, \beta)$ be given. Then there are a lift $\tilde{\gamma}$ of γ, a lift $\tilde{\alpha}$ of α and n lifts $\tilde{\beta}_1 \ldots \tilde{\beta}_n$ of $\mathcal{B}$ satisfying (1)–(5) of Definition 4.2.2. If a lift $\tilde{\alpha}'$ of $\mathcal{A}_1$ intersects both $\tilde{\gamma}$ and some $\tilde{\beta}_i$, then $\tilde{\alpha}' = \tilde{\alpha}$ because $\tilde{\alpha}'$ is a lift of $\mathcal{A}_2$. So (5) of Definition 4.2.2 holds. Note that the conditions (1)–(4) of Definition 4.2.2 are satisfied immediately with $\tilde{\gamma}$, $\tilde{\alpha}$ and $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$. Then γ is contained in $\mathrm{PP}_n(\mathcal{A}_1, \alpha, \mathcal{B}, \beta)$. Because γ is an arbitrary simple closed geodesic in $\mathrm{PP}_n(\mathcal{A}_2, \alpha, \mathcal{B}, \beta)$, we have $\mathrm{PP}_n(\mathcal{A}_2, \alpha, \mathcal{B}, \beta) \subseteq \mathrm{PP}_n(\mathcal{A}_1, \alpha, \mathcal{B}, \beta)$.

(4) Let $\gamma \in \mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_1, \beta)$ be given. By definition, some lift $\tilde{\alpha}$ of α, lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\mathcal{B}_1$ and a lift $\tilde{\gamma}$ of γ exist satisfying the conditions (1)–(5) of Definition 4.2.2. Because $\beta_1, \ldots, \beta_n$ are also lifts of $\mathcal{B}_2$, the simple closed geodesic γ is also contained in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_2, \beta)$. Therefore, we have $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_1, \beta) \subseteq \mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}_2, \beta)$. □

Lemma 4.2.4. *Let α and β be simple closed geodesics on Σ intersecting each other transversally. If a simple closed geodesic γ on Σ is contained in $\mathrm{PP}_n(\{\alpha\}, \alpha, \{\beta\}, \beta)$ for some n, then we have $n \leq \mathrm{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\}$.*

Proof. Assume that $\gamma \in \mathrm{PP}_n(\{\alpha\}, \alpha, \beta)$ for some $n > 2$. Then there are a lift $\tilde{\gamma}$ of γ, a lift $\tilde{\alpha}$ of α and n lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of β such that $\tilde{\alpha}$, $\tilde{\beta}_i$ and $\tilde{\gamma}$ cross each other for each i by 4 of Definition 4.2.2. Then $n \leq \operatorname{lcm}\{i(\alpha,\beta), i(\beta,\gamma)\}$ by Proposition 3.2.5(3). $\square$

Proposition 4.2.5. *Let α, β and γ be simple closed geodesics on Σ such that β intersects both α and γ transversally. If n is more than 2 and an integer m satisfies the inequality*

$$|m| \geq \frac{n + 2 \cdot i(\alpha,\gamma)}{i(\beta,\gamma)} + 2,$$

then for every multicurve $\mathcal{B}$ containing β,

$$T_\beta^m\left(\{\alpha\} \cup \mathrm{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)\right) \subseteq \mathrm{PP}_n(\mathcal{B}, \beta, \{\gamma\}, \gamma).$$

Proof. Fix a multicurve $\mathcal{B}$ which contains β. Let δ be either α or a simple closed geodesic in $\mathrm{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)$. Let $\tilde{\delta}$ and $\tilde{\alpha}$ be lifts of δ and α, respectively, and let $\tilde{\beta}_{-1}$, $\tilde{\beta}_0$ and $\tilde{\beta}_1$ be lifts of $\mathcal{B}$ satisfying the following.

- If δ is equal to α, we choose an arbitrary lift $\tilde{\alpha}$ of α. And set $\tilde{\delta} = \tilde{\alpha}$. Let $\tilde{\beta}_0$ be a lift of β crossing $\tilde{\alpha}$. Let $\tilde{\beta}_{-1}$ and $\tilde{\beta}_1$ be lifts of $\mathcal{B}$ crossing $\tilde{\alpha}$ such that $\tilde{\beta}_0$ is the unique lift of $\mathcal{B}$ separating $\tilde{\beta}_{-1}$ from $\tilde{\beta}_1$.

- If δ is contained in $PP_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)$, then there are lifts $\tilde{\delta}$ and $\tilde{\alpha}$ of δ and α, respectively, such that $\tilde{\delta}$ intersects $\tilde{\alpha}$ transversally. Let $\tilde{\beta}_{-1}, \tilde{\beta}_0$ and $\tilde{\beta}_1$ be lifts of $\mathcal{B}$ satisfying (2)–(5) of Definition 4.2.2. In addition, we add the condition that $\tilde{\beta}_0$ is a unique lift of $\mathcal{B}$ which separates $\tilde{\beta}_{-1}$ from $\tilde{\beta}_1$.

We divide the proof into two claims.

Claim 1: The number of lifts of γ separating $\tilde{\beta}_{-1}$ from $\tilde{\beta}_1$ is at most $2 \cdot i(\alpha,\gamma)$.

Let h be a primitive element of $\pi_1(\Sigma)$ which preserves $\tilde{\alpha}$ and moves $\tilde{\beta}_{-1}$ into the connected component of $\mathbb{H}^2 \setminus \tilde{\beta}_{-1}$ containing $\tilde{\beta}_0$. Because there is no lift of $\mathcal{B}$ separating $\tilde{\beta}_0$ from $\tilde{\beta}_{-1}$, it holds that $h\tilde{\beta}_{-1}$ cannot separate $\tilde{\beta}_0$ from $\tilde{\beta}_{-1}$. Then either $h\tilde{\beta}_{-1} = \tilde{\beta}_0$ or $\tilde{\beta}_0$ separates $\tilde{\beta}_{-1}$ from $h\tilde{\beta}_{-1}$. Similarly, since h moves $\tilde{\beta}_0$ into the connected component of $\mathbb{H}^2 \setminus \tilde{\beta}_0$ containing $\tilde{\beta}_1$, either $h\tilde{\beta}_0$ is equal to $\tilde{\beta}_1$ or $\tilde{\beta}_1$ separates $\tilde{\beta}_0$ from $h\tilde{\beta}_0$. Combining these facts, we have either $h^2\tilde{\beta}_{-1}$ is equal to $\tilde{\beta}_1$ or $\tilde{\beta}_1$ separates $\tilde{\beta}_{-1}$ from $h^2\tilde{\beta}_{-1}$.

Let $\Phi_\gamma := \Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 3.1.3 for some sufficiently small number

$\epsilon > 0$. For each $i \in \{-1, 0, 1\}$, we have $\Phi_\gamma(\tilde{\alpha})$ intersects $\Phi_\gamma(\tilde{\beta}_i)$ because $\tilde{\alpha}$ intersects $\tilde{\beta}_i$ transversally. Every path joining $\Phi_\gamma(\tilde{\beta}_{-1})$ and $\sigma_\gamma(h^2)\Phi_\gamma(\tilde{\beta}_{-1})$ intersects $\Phi_\gamma(\tilde{\beta}_1)$ by the above. So it is satisfied that $d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \Phi_\gamma(\tilde{\beta}_1)) \le d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \sigma_\gamma(h^2)\Phi_\gamma(\tilde{\beta}_{-1}))$.

Because of the median property of the dual tree $\mathcal{Y}_\gamma$, the shortest geodesic segment between $\Phi_\gamma(\tilde{\beta}_{-1})$ and $\Phi_\gamma(\tilde{\beta}_1)$ is a subset of $\Phi_\gamma(\tilde{\alpha})$. Since the translation length of h on $\mathcal{Y}_\gamma$ is $i(\alpha, \gamma)$ by Lemma 3.2.2, it is satisfied that

$$d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \Phi_\gamma(\tilde{\beta}_1)) \le 2 \cdot i(\alpha, \gamma).$$

So the claim holds.

Choose a vertex of the dual tree $\mathcal{Y}_\beta$ of β on the geodesic segment connecting $\Phi_\beta(\tilde{\beta}_{-1})$ and $\Phi_\beta(\tilde{\beta}_0)$. Let $\tilde{T}_\beta$ be the lift of a Dehn twist T_β which fixes $\mathrm{st}(v)$. (*cf.* Proposition 4.1.2) Because the midpoints $\Phi_\beta(\tilde{\beta}_{-1})$ and $\Phi_\beta(\tilde{\beta}_0)$ are contained in $\mathrm{st}(v)$, we have $\tilde{T}_\beta^m \tilde{\beta}_{-1} = \tilde{\beta}_{-1}$ and $\tilde{T}_\beta^m \tilde{\beta}_0 = \tilde{\beta}_0$. Meanwhile, because $\tilde{\beta}_0$ is the unique lift of β separating $\tilde{\beta}_1$ from $\Phi_\beta^{-1}(v)$, it holds that $\tilde{T}_\beta^m \tilde{\beta}_1 = h_0^m \tilde{\beta}_1$ for some primitive element h_0 preserving $\tilde{\beta}_0$ by Proposition 4.1.2.

Claim 2: The number of lifts of γ separating $\tilde{\beta}_{-1}$ from $\tilde{T}_\beta^m \tilde{\beta}_1$ is at most n.

If $\mathrm{pr} : \mathcal{Y}_\gamma \to \Phi_\gamma(\tilde{\beta}_0)$ be the closest-point projection to $\Phi_\gamma(\tilde{\beta}_0)$, we write P_i as the projective image $\mathrm{pr}(\Phi_\gamma(\tilde{\beta}_i))$ for each $i = -1, 1$. Then the length $l(P_i)$ of each P_i is at most $i(\beta, \gamma)$ by Proposition 3.2.5. Let w' and w'' be the vertices of P_1 such that $d_\gamma(w', \sigma_\gamma(h_0^m)w'') = d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1)$. Because $\Phi_\gamma(\tilde{\beta}_0)$ is preserved by h_0,

$$\begin{aligned} d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1) &= d_\gamma(w', \sigma_\gamma(h_0^m)w'') \\ &\ge d_\gamma(w', \sigma_\gamma(h_0^m)w') - d_\gamma(\sigma_\gamma(h_0^m)w', \sigma_\gamma(h_0^m)w'') \\ &\ge |m| \cdot \mathrm{tr}_\gamma\, h_0 - l(P_1) \\ &\ge (|m| - 1) \cdot i(\beta, \gamma). \end{aligned}$$

By the hypothesis and the previous claim,

$$\begin{aligned} d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \sigma_\gamma(h_0^m)\Phi_\gamma(\tilde{\beta}_1)) &\ge d_\gamma(P_{-1}, \sigma_\gamma(h_0^m)P_1) \\ &\ge d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1) - d_\gamma(P_{-1}, P_1) - l(P_{-1}) \\ &\ge (|m| - 1) \cdot i(\beta, \gamma) - 2 \cdot i(\alpha, \gamma) - i(\beta, \gamma) \\ &\ge n. \end{aligned}$$

So there are at least n edges of $\mathcal{Y}_\gamma$ between $\Phi_\gamma(\tilde{\beta}_{-1})$ and $\sigma_\gamma(h_0^m)\Phi_\gamma(\tilde{\beta}_1)$. In other words, there are n lifts $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ of γ separating $\tilde{\beta}_{-1}$ from $h_0^m\tilde{\beta}_1$. We proved the claim.

Since $\tilde{T}_\beta^m\tilde{\delta}$ intersects both $\tilde{\beta}_{-1}$ and $\tilde{T}_\beta^m\tilde{\beta}_1$ transversally, it also intersects all $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ transversally. Note that $\tilde{T}_\beta^m\tilde{\beta}_0$ is the unique lift of $\mathcal{B}$ separating $\tilde{\beta}_{-1}$ from $\tilde{T}_\beta^m\tilde{\beta}_1$. If another lift $\tilde{\beta}''$ of $\mathcal{B}$ intersects both $\tilde{T}_\beta^m\tilde{\delta}$ and $\tilde{\gamma}_i$ for some i, then it must separate $\tilde{\beta}_{-1}$ from $h_0^m\tilde{\beta}_1$ so that $\tilde{\beta}'' = \tilde{\beta}_0$. Hence, $T_\beta^m\delta$ is contained in $\mathrm{PP}_n(\mathcal{B}, \beta, \{\gamma\}, \gamma)$. □

Proposition 4.2.6. *If $\mathcal{A}$ and $\mathcal{B}$ are multicurves and simple closed geodesics $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ intersect each other transversally, then*

$$T_\gamma^m(\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)) \subset \mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$$

for all $\gamma \in \mathcal{A} - \{\alpha\}$, $n > 2$ and $m \in \mathbb{Z}$.

Proof. Consider a simple closed geodesic δ in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$. Let $\tilde{\delta}$ and $\tilde{\alpha}$ be lifts of δ and α, respectively, and let $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ be lifts of $\mathcal{B}$ satisfying the conditions of Definition 4.2.2. Let $\tilde{\alpha}_{-1}$ and $\tilde{\alpha}_1$ be lifts of $\mathcal{A}$ such that $\tilde{\alpha}$ is the unique lift of $\mathcal{A}$ separating $\tilde{\alpha}_{-1}$ from $\tilde{\alpha}_1$. Because γ is disjoint from α, there is a lift $\tilde{T}_\gamma$ of a Dehn twist T_γ which fixes $\tilde{\alpha}$ pointwise.

That is, $\tilde{T}_\gamma^m\tilde{\alpha}_i = \tilde{\alpha}_i$ for all $i = -1, 1$. So $\tilde{T}_\gamma^m\tilde{\delta}$ still intersects both $\tilde{\alpha}_{-1}$ and $\tilde{\alpha}_1$ transversally. It implies that all $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ intersect $\tilde{T}_\gamma^m\tilde{\delta}$ transversally. Since $\tilde{\alpha}$ is the unique lift of $\mathcal{A}$ separating $\tilde{\alpha}_{-1}$ from $\tilde{\alpha}_1$, (5) of Definition 4.2.2 also holds for $\tilde{T}_\gamma\tilde{\beta}$. As a result, $T_\gamma^m\delta$ is contained in $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$. Therefore, $T_\gamma^m(\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta))$ is a subset of $\mathrm{PP}_n(\mathcal{A}, \alpha, \mathcal{B}, \beta)$. □

Theorem 4.2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be multicurves on Σ, and let α and β be simple closed geodesics in $\mathcal{A}$ and $\mathcal{B}$, respectively. Assume that α intersects β transversally. Then the following hold.*

1. *If a simple closed geodesic γ is contained in $\mathrm{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta)$ for some $n > 2$, then n is at most $\mathrm{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\}$.*

2. *Let γ be a simple closed geodesic intersecting β transversally. If n and m are positive integers satisfying that $n > 2$ and*

$$|m| \geq \frac{n + 2 \cdot i(\alpha, \gamma)}{i(\beta, \gamma)} + 2,$$

then

$$T_\beta^m\left(\{\alpha\} \cup \mathrm{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta)\right) \subseteq \mathrm{PP}_n(\mathcal{B}, \beta, \{\gamma\}, \gamma).$$

3. *If γ is a simple closed geodesic in $\mathcal{A}\backslash\{\alpha\}$ and m is an arbitrary integer, then it is satisfied that*

$$T_\gamma^m(\mathrm{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta)) \subseteq \mathrm{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta).$$

Proof. (1) By Lemma 4.2.3(3), we have

$$\mathrm{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta) \subseteq \mathrm{PP}_n(\{\alpha\}, \alpha, \{\beta\}, \beta).$$

By Definition 4.2.2(4), there are two lifts $\tilde{\gamma}$ and $\tilde{\alpha}$ of γ and α, respectively, such that intersect at least n lifts of β simultaneously. So Proposition 3.2.5(3) implies that $n \leq \Delta_\beta(\tilde{\alpha}, \tilde{\gamma}) \leq \mathrm{lcm}\{i(\alpha, \gamma), i(\beta, \gamma)\}$.

(3) It is an immediate result of Proposition 4.2.6.

(2) By Lemma 4.2.3(2), (3) and (4), we have

$$\mathrm{PP}_n(\mathcal{A}, \alpha, \beta) \subseteq \mathrm{PP}_3(\{\alpha\}, \alpha, \beta) \subseteq \mathrm{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta).$$

Therefore, $T_\beta^m(\{\alpha\} \cup \mathrm{PP}_n(\mathcal{A}, \alpha, \beta)) \subseteq \mathrm{PP}_n(\mathcal{B}, \beta, \gamma)$ by Proposition 4.2.5. □

4.3 Right-angled Artin groups generated by powers of Dehn twists

Definition 4.3.1. Let $\Gamma = (V, E)$ be a simplicial graph where V is the vertex set of Γ and $E \subset \{\{v, w\} \subset V \mid v \neq w\}$ is the edge set of Γ. Then the *right-angled Artin group* , denoted by $A(\Gamma)$, is a group with the presentation

$$A(\Gamma) = \langle V \mid [u, v] = uvu^{-1}v^{-1} = 1 \text{ for all } \{u, v\} \in E \rangle.$$

The motivation of this section is Koberda's theorem.

Theorem 4.3.2 (Theorem 1.1, Koberda [24])**.** *Let $\mathcal{G} \subset \mathrm{Mod}(\Sigma)$ be a finite set of Dehn twists on simple closed curves or purely pseudo-Anosov mapping classes on connected subsurfaces. If $\mathcal{G}$ is irredudant, then there is a positive integer N such that the subgroup generated by $\{f^n \mid f \in \mathcal{G}\}$ is isomorphic to a right-angled Artin group for all $n \geq N$.*

Our main theorem is the following.

Theorem 4.3.3. *Let $\mathcal{F}$ be a set of finitely many simple closed geodesics on Σ. Let Γ be the simplicial graph whose vertex set is $\mathcal{F}$ satisfying that two vertices are joined by an edge if and only if they are disjoint as simple closed geodesics. Write $\Delta(\mathcal{F}) := \max\{\mathrm{lcm}\{i(\alpha,\beta), i(\beta,\gamma)\} \mid \alpha,\beta,\gamma \in \mathcal{F}\}$. For an integer m, let φ be the homomorphism from $A(\Gamma)$ to $\langle\{T_\gamma^m \mid \gamma \in \mathcal{F}\}\rangle$ sending each alphabet γ to T_γ^m.*

1. *If $\Delta(\mathcal{F}) = 1$ and $|m| \geq 7$, then φ is an isomorphism.*
2. *If $\Delta(\mathcal{F}) \geq 2$ and m is an integer satisfying that*

$$|m| \geq \max_{\substack{\alpha,\beta,\gamma\in\mathcal{F},\\ i(\beta,\gamma)>0}} \frac{\Delta(\mathcal{F}) + 2\cdot i(\alpha,\gamma)+1}{i(\beta,\gamma)} + 2,$$

then φ is an isomorphism.

A reduced word $W = \gamma_n^{k_n}\dots\gamma_1^{k_1}$ of $A(\Gamma)$ is called a *central word* if $\gamma_i \neq \gamma_j$ and γ_i commutes with γ_j for all distinct $i,j \in \{1,\dots,n\}$. And the *central form* of a reduced word W of $A(\Gamma)$ is a decomposition of W into the product of central words $W_1,\dots,W_n$ such that $W = W_n\dots W_1$ and the last alphabet of W_i does not commute with the last alphabet of W_{i+1} for every $i \in \{1,\dots,n-1\}$. Note that every nonidentity element of $A(\Gamma)$ admits a word of central form.

Proof of Theorem 4.3.3. First, we assume that Γ is the join of some nonempty subgraphs Γ_1 and Γ_2. (The *join* of two graphs $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ is the graph $(V_1\cup V_2, E_1\cup E_2\cup\{\{u,v\}\mid u\in V_1, u\in V_2\})$.) Then $A(\Gamma)\cong A(\Gamma_1)\times A(\Gamma_2)$ by the definition of a right-angled Artin group. In this case, we can divide the proof into the cases for Γ_1 and Γ_2. That is, if we show that the restriction of φ to $A(\Gamma_1)$ and $A(\Gamma_2)$ is an isomorphism, then it implies that φ is also an isomorphism.

So we need only to prove the simplest case that Γ cannot be decomposed into the join of nonempty subgraphs. It means that every simple closed geodesic in $\mathcal{F}$ intersects another simple closed geodesic of $\mathcal{F}$ transversally.

Let $W = W_n\dots W_1 \in A(\Gamma)$ be an arbitrary nonidentity reduced word of central form. We will show that $\varphi(W)$ acts nontrivially on the set of simple closed geodesics. For each $i = 1,\dots,n$, let $\mathrm{supp}(W_i)$ be the subset of $\mathcal{F}$ containing all alphabets of W_i. Because W_i is a central word, the set $\mathrm{supp}(W_i)$ forms a multicurve. For each $i \in \{1, \dots, n\}$, let γ_i be the last alphabet of W_i. And choose an arbitrary simple closed geodesic γ_{n+1} in $\mathcal{F}$ intersecting γ_n transversally. Write α as γ_2.

Let

$$N := \begin{cases} \Delta(\mathcal{F}) + 1 & \text{if } \Delta(\mathcal{F}) \geq 2, \text{ or} \\ 3 & \text{if } \Delta(\mathcal{F}) = 1. \end{cases}$$

Write $W_1 = \gamma_{1,j_1}^{k_{1,j_1}} \gamma_{1,j_1-1}^{k_{1,j_1-1}} \dots \gamma_{1,1}^{k_{1,1}} \gamma_1^{k_1}$ for some $\gamma_{1,1} \dots \gamma_{1,j_1} \in \mathrm{supp}(W_1)$ and $k_1, k_{1,1}, \dots, k_{1,j_1} \in \mathbb{Z}$. Then it holds that

$$\varphi(\gamma_1^{k_1})\alpha = T_{\gamma_1}^{mk_1}(\alpha) \in \mathrm{PP}_N(\mathrm{supp}(W_1), \gamma_1, \{\gamma_2\}, \gamma_2)$$

by Theorem 4.2.7(2). Applying Theorem 4.2.7(3) repeatedly, $\varphi(\gamma_{1,j_1}^{k_{1,j_1}} \dots \gamma_{1,1}^{k_{1,1}})$ sends $\varphi(\gamma_1^{k_1})\alpha$ into $\mathrm{PP}_N(\mathrm{supp}(W_1), \gamma_1, \{\gamma_2\}, \gamma_2)$ again. Consequently, $\varphi(W_1)\alpha$ is contained in $\mathrm{PP}_N(\mathrm{supp}(W_1), \gamma_1, \{\gamma_2\}, \gamma_2)$.

Suppose that the simple closed geodesic $\varphi(W_{l-1}W_{l-2} \dots W_1)\alpha$, denoted by α_{l-1}, is an element of $\mathrm{PP}_N(\mathrm{supp}(W_{l-1}), \gamma_{l-1}, \{\gamma_l\}, \gamma_l)$ for some $l \in \{2, \dots, n\}$. Write W_l by $\gamma_{l,j_l}^{k_{l,j_l}} \dots \gamma_{l,1}^{k_{l,1}} \gamma_l^{k_l}$ for some simple closed geodesics $\gamma_{l,1}, \dots, \gamma_{l,j_l} \in \mathrm{supp}(W_l)$ and integers $k_l, k_{l,1}, \dots, k_{l,j_l} \in \mathbb{Z}$. Then

$$\varphi(\gamma_l^{k_l})\alpha_{l-1} \in \mathrm{PP}_N(\mathrm{supp}(W_l), \gamma_l, \{\gamma_{l+1}\}, \gamma_{l+1})$$

by Theorem 4.2.7(2). And by Theorem 4.2.7(3), we have

$$\varphi(W_l)\alpha_{l-1} = \varphi(\gamma_{l,j_l}^{k_{l,j_l}} \dots \gamma_{l,1}^{k_{l,1}})\varphi(\gamma_l^{k_l})\alpha_{l-1} \in \mathrm{PP}_N(\mathrm{supp}(W_l), \gamma_l, \{\gamma_{l+1}\}, \gamma_{l+1}).$$

By induction, it is satisfied that $\varphi(W)\alpha \in \mathrm{PP}_N(\mathrm{supp}(W_n), \gamma_n, \{\gamma_{n+1}\}, \gamma_{n+1})$.

By Theorem 4.2.7(1), the simple closed geodesic α is not contained in $\mathrm{PP}_N(\mathrm{supp}(W_n), \gamma_n, \{\gamma_{n+1}\}, \gamma_{n+1})$ So the action of $\varphi(W)$ on the set of simple closed geodesics is nontrivial. Since W is an arbitrary nonidentity reduced word of $A(\Gamma)$, the action of $A(\Gamma)$ on the set of simple closed geodesics is faithful. Therefore, φ is a monomorphism, i.e., it is an isomorphism. $\square$

Corollary 4.3.4. *Let $\mathcal{F}$ be a finite set of simple closed geodesics on Σ. Assume that $\{i(\alpha, \beta) \mid \alpha, \beta \in \mathcal{F}\} \subseteq \{0, N\}$ for some $N \geq 2$. Then the group generated by $\{T_\gamma^6 \mid \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group.*

Proof. Suppose that $\mathcal{F}$ is not a multicurve. Then $\Delta(\mathcal{F}) = N$ and

$$\max_{\alpha,\beta,\gamma\in\mathcal{F},\beta\pitchfork\gamma} \frac{\Delta(\mathcal{F}) + 2\cdot i(\alpha,\gamma) + 1}{i(\beta,\gamma)} + 2 \leq 5 + \frac{1}{N}.$$

By Theorem 4.3.3, the group generated by $\{T_\gamma^6 \,|\, \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group. $\square$

Corollary 4.3.5. *Let $\mathcal{F}$ be a finite set of simple closed geodesics on Σ. If N is the maximum among all intersection numbers of pairs of simple closed geodesics in $\mathcal{F}$, then the group generated by $\{T_\gamma^m \,|\, \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group for all integers $m \geq N^2 + N + 3$.*

Proof. Assume that $\mathcal{F}$ is not a multicurve. Since $\Delta(\mathcal{F}) \leq N(N-1)$, it holds that

$$\begin{aligned}\max_{\alpha,\beta,\gamma\in\mathcal{F},\beta\pitchfork\gamma} \frac{\Delta(\mathcal{F}) + 2\cdot i(\alpha,\gamma) + 1}{i(\beta,\gamma)} + 2 &\leq \frac{N(N-1) + 2N + 1}{1} + 2 \\ &= N^2 + N + 3.\end{aligned}$$

Therefore, by Theorem 4.3.3, the group generated by $\{T_\gamma^m \,|\, \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group for all $m \geq N^2 + N + 3$. $\square$

Part II

Homomorphisms from $\mathrm{Aut}(\pi_1(\Sigma))$ into the quasi-isometry groups of $\mathrm{CAT}(0)$ cube complexes

Chapter 5

Preliminaries for CAT(0) cube complexes

5.1 Cube complexes and Gromov's characterization

Definition 5.1.1. A *cube* or an *n-cube* is a Euclidean polyhedral cell which is isometric to the subspace $[0,1]^n \subset \mathbb{R}^n$, the product of n closed unit intervals. We usually say a *vertex*, an *edge* and a *square*, instead of a 0-cube, a 1-cube and a 2-cube.

Definition 5.1.2. A *cube complex* is a polyhedral complex whose cells are cubes. For a cube complex X, the n-skeleton of X is denoted by $X^{(n)}$. A cube complex is *finite-dimensional* if the dimension of every cube of the complex is uniformly bounded. A CAT(0) *cube complex* is a cube complex which is a CAT(0) space.

Remark 5.1.3. The metrics of each cube of a cube complex define a quotient pseudo-metric of X.[1] This quotient pseudo-metric of X is a metric if X is finite-dimensional.

Definition 5.1.4. Let x be a vertex of a finite-dimensional cube complex X. The *link* of x, written by $lk(x)$, is the boundary of the 1/3-neighborhood of x.

[1] See [7, I.5.19] for how to define a quotient pseudo-metric from the metrics of cubes.

Remark 5.1.5. Let X be a cube complex.

1. For a cube $C \subset X$ and a vertex $x \in C$, $lk(x) \cap C$ is a right-angled spherical simplex.[2] Nonetheless, it does not guarantee the link $lk(x)$ is simplicial. There could exist a loop-edge or a multi-edge in $lk(x)$.
2. If x is a vertex of X and $lk(x)$ is simplicial, then $lk(x)$ is a piecewise right-angled spherical complex.[3]

Definition 5.1.6. A cube complex is called *locally finite* if the link of every vertex is compact.

Definition 5.1.7. A simplicial complex is *flag* if every vertices connected pairwise by edges form a simplex.

Lemma 5.1.8 ([19], Theorem II.5.18 in [7])**.** *Let Y be a finite-dimensional piecewise right-angled spherical simplicial complex. Then, Y is a* CAT(1) *space if and only if Y is a flag complex.*

The following proposition is Gromov's charaterization for CAT(0) cube complexes. For more details, see [19] and [7, Theorem II.5.20].

Proposition 5.1.9 (Gromov's charaterization)**.** *Let X be a finite-dimensional cube complex. Then the following two statements are equivalent.*

1. *X is a locally* CAT(0) *space.*
2. *The link of every vertex of X is a flag complex.*

In particular, the following are also equivalent by the Cartan-Hadamard theorem [7, Theorem II.4.1].

I. *X is a* CAT(0) *space.*

II. *X is connected and simply connected, and the link of every vertex of X is a flag complex.*

Definition 5.1.10. A finite-dimensional cube complex is called a *nonpositively curved cube complex* or an *NPC cube complex* if the link of every vertex is a flag complex.

[2] Every vertex angle in $lk(x) \cap C$ is $\pi/2$.

[3] A piecewise right-angled spherical complex is also called an *all-right spherical complex* in [7, Definition II.5.17].

Remark 5.1.11.

1. By Proposition 5.1.9 and Definition 5.1.10, a finite-dimensional connected cube complex X is a CAT(0) cube complex if and only if X is a simply connected NPC cube complex.

2. Leary [27] showed that Proposition 5.1.9 also holds for infinite-dimensional cube complexes.

Example 1. Note that every topological graph is homeomorphic to some 1-dimensional cube complex. Let X be a 1-dimensional cube complex. Then the link of every vertex of X does not have an edge so that it is a flag complex. Hence, X is an NPC cube complex.

Then, which topological graphs are CAT(0) cube complexes? Note that every connected simply connected graph is a tree. By Gromov's characterization, a 1-dimensional cube complex is CAT(0) if and only if it is homeomorphic to a tree.

Corollary 5.1.12. *Let X be a topological space.*

1. *The space X is a topological graph if and only if X is homeomorphic to a 1-dimensional NPC cube complex.*

2. *The space X is a tree if and only if X is homeomorphic to a 1-dimensional* CAT(0) *cube complex.*

5.2 Hyperplanes and halfspaces

Definition 5.2.1 (Parallelism)**.** Let X be an NPC cube complex. The *parallelism* is the equivalence relation, on the set of all edges of X, which is generated by the following statement. *Two edges e_1 and e_2 are parallel if a square of X contains e_1 and e_2 as opposite sides.*

Definition 5.2.2 (Midcube)**.** Let C be an n-cube and $\phi : [0,1]^n \to C$ an isometry. A subset M of C is called a *midcube* if there is $i \in \{1, \dots, n\}$ such that $M = \phi(p_i^{-1}(1/2))$ where $p_i : [0,1]^n \to [0,1]$ is the projection to the i-th coordinate.

Remark 5.2.3. Let X be an NPC cube complex and C an n-cube in X for some $n > 0$. Choose a midcube M of C.

1. The dimension of M is $n - 1$.

2. The number of midcubes contained in C is exactly n. In particular, an edge of C intersects a unique midcube at the midpoint of the edge.
3. Each face of M is a midcube of a face of C. Precisely, if a face F of C intersects M, then $F \cap M$ is a face of M.

Remark 5.2.4. Let X be an NPC cube complex, and let e_1 and e_2 be edges in a square S of X. Then, e_1 is parallel to e_2 if and only if some midcube of S intersects both e_1 and e_2. So the following statements are equivalent.

1. Two edges e_1 and e_2 in X are parallel.
2. There exist finitely many 1-midcubes in X such that their union is homeomorphic to a line segment and the endpoints of the union are the midpoints of e_1 and e_2.

Definition 5.2.5 (Hyperplane)**.** Let X be an NPC cube complex, and let $[e]$ a parallelism class of edges. The *hyperplane* corresponding to $[e]$ is the union of midcubes whose vertices are midpoints of edges contained in $[e]$.

Definition 5.2.6 (Duality)**.** An edge e is said to be *dual* to a hyperplane $\hat{\mathfrak{h}}$ or a hyperplane $\hat{\mathfrak{h}}$ is said to be *dual* to an edge e if the midpoint of e is a vertex of $\hat{\mathfrak{h}}$.

Definition 5.2.7 (Transversality)**.** On an NPC cube complex, a hyperplane $\hat{\mathfrak{h}}_1$ is *transverse* to a hyperplane $\hat{\mathfrak{h}}_2$ if $\hat{\mathfrak{h}}_1 \neq \hat{\mathfrak{h}}_2$ and they intersect each other.[4]

Definition 5.2.8 (Osculation)**.** Let $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ be hyperplanes in an NPC cube complex. We say that $\hat{\mathfrak{h}}_1$ *osculates* $\hat{\mathfrak{h}}_2$ if there are two edges e_1 and e_2, which are dual to $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$, respectively, such that e_1 and e_2 share a vertex but they are not in the same square.

Definition 5.2.9 (Halfspace)**.** Let $\tilde{X}$ be a CAT(0) cube complex and $\hat{\mathfrak{h}}$ a hyperplane in $\tilde{X}$. Then the maximal subcomplex on a connected component of $\tilde{X} \setminus \hat{\mathfrak{h}}$ is called a *halfspace* of $\hat{\mathfrak{h}}$. We say that two halfspaces in a CAT(0) cube complex are *transverse* to each other if their hyperplanes cross each other.

Remark 5.2.10. Let $\tilde{X}$ be a CAT(0) cube complex.

[4] Or, we say that $\hat{\mathfrak{h}}_1$ *crosses* $\hat{\mathfrak{h}}_2$.

1. If $\hat{\mathfrak{h}}$ is a hyperplane in $\tilde{X}$, we write $\mathfrak{h}$ as one of the halfspaces of $\hat{\mathfrak{h}}$. Conversely, for a halfspace $\mathfrak{h}$, the hyperplane of $\mathfrak{h}$ is denoted by $\hat{\mathfrak{h}}$.

2. If $\mathcal{H}$ is the set of all halfspaces of $\tilde{X}$ and $\hat{\mathcal{H}}$ is the set of all hyperplanes in $\tilde{X}$, then the map $\wedge : \mathcal{H} \to \hat{\mathcal{H}}$, which sends each halfspace $\mathfrak{h}$ to the hyperplane $\hat{\mathfrak{h}}$ of $\mathfrak{h}$, is a two-to-one correspondence between $\mathcal{H}$ and $\hat{\mathcal{H}}$.

Definition 5.2.11. Let $\tilde{X}$ be a CAT(0) cube complex and $\mathcal{H}$ the set of all halfspaces in $\tilde{X}$. Define an operator $* : \mathcal{H} \to \mathcal{H}$ with $\mathfrak{h} \mapsto \mathfrak{h}^*$ satisfying that, for all $\mathfrak{h} \in \mathcal{H}$, the halfspace $\mathfrak{h}$ is disjoint from $\mathfrak{h}^*$ and their hyperplanes are equal. We call $*$ the *opposite operator* on $\mathcal{H}$. For each $\mathfrak{h} \in \mathcal{H}$, we say $\mathfrak{h}^*$ is the *opposite halfspace* of $\mathfrak{h}$.

Remark 5.2.12. Let $\tilde{X}$ be a CAT(0) cube complex. Let $\hat{\mathfrak{h}}$ be a hyperplane of $\tilde{X}$, and let $\mathfrak{h}$ and $\mathfrak{h}^*$ be the pair of halfspaces of $\hat{\mathfrak{h}}$. Since $\hat{\mathfrak{h}}$ does not contain any vertex of $\tilde{X}$, each vertex of $\tilde{X}$ belongs to either $\mathfrak{h}$ or $\mathfrak{h}^*$ by the maximality of halfspaces. In general, a cube C of $\tilde{X}$ does not intersect $\hat{\mathfrak{h}}$ if and only if C is contained in either $\mathfrak{h}$ or $\mathfrak{h}^*$.

Definition 5.2.13. Let A and B be subcomplexes of a CAT(0) cube complex and $\hat{\mathfrak{h}}$ a hyperplane. We say that $\hat{\mathfrak{h}}$ *separates* A and B if there is a halfspace $\mathfrak{h}$ of $\hat{\mathfrak{h}}$ such that $A \subseteq \mathfrak{h}$ and $B \subseteq \mathfrak{h}^*$.

Lemma 5.2.14 (Lemma 3.6 in [41])**.** *Any pair of hyperplanes in a* CAT(0) *cube complex do not inter-osculate. That is, they cannot cross and osculate simultaneously.*

Proposition 5.2.15 (Theorem 4.10, 4.11, 4.14 in [35], Theorem 1.1 in [36])**.** *Let $\tilde{X}$ be a* CAT(0) *cube complex and $\hat{\mathfrak{h}}$ a hyperplane in $\tilde{X}$.*

1. *The hyperplane $\hat{\mathfrak{h}}$ does not cross itself.*

2. *The complement $\tilde{X} \setminus \hat{\mathfrak{h}}$ is the disjoint union of exactly two connected components.*

3. *The hyperplane $\hat{\mathfrak{h}}$ is a* CAT(0) *cube complex.*

4. *The intersection of pairwise transverse hyperplanes is nonempty.*

Definition 5.2.16. Let $\hat{\mathfrak{h}}$ be a hyperplane in an NPC cube complex. The *carrier* of $\hat{\mathfrak{h}}$, denoted by $\mathcal{N}(\hat{\mathfrak{h}})$, is the union of cubes which contains edges dual to $\hat{\mathfrak{h}}$.

Definition 5.2.17. Let X and Y be cube complexes and $\phi : X \to Y$ a continuous map.

1. The map ϕ is called a *cubical map* if ϕ sends each cube in X to a cube in Y.
2. The map ϕ is called an *(cubical) embedding* if ϕ is an isometric embedding and a cubical map.
3. The map ϕ is called an *(cubical) isomorphism* if ϕ is a cubical map and an isometry.
4. The cube complex X is said to be *isomorphic* to Y if there is an isomorphism between X and Y.

Lemma 5.2.18. *Let $\tilde{X}$ be a* CAT(0) *cube complex and $\mathfrak{h}$ a halfspace of a hyperplane $\hat{\mathfrak{h}}$.*

1. *The carrier $\mathcal{N}(\hat{\mathfrak{h}})$ is isomorphic to $\hat{\mathfrak{h}} \times [0,1]$.*
2. *The intersection $\mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}$ is isomorphic to $\hat{\mathfrak{h}}$.*

To distinguish the intersection of a hyperplanes and a subcomplex from the intersection of two subcomplexes, we use another term for the intersetion of a hyperplane and a subcomplex.

Definition 5.2.19. Let Y be a subcomplex of $\tilde{X}$. We say a hyperplane $\hat{\mathfrak{h}}$ *passes through* Y if $\hat{\mathfrak{h}}$ intersects Y.

Lemma 5.2.20. *Let $\tilde{X}$ be a* CAT(0) *cube complex and Y a connected subcomplex of $\tilde{X}$. If $\hat{\mathfrak{h}}$ is a hyperplane and $\mathfrak{h}$ is a halfspace of $\hat{\mathfrak{h}}$, then the followings are equivalent.*

1. *$\hat{\mathfrak{h}}$ passes through Y.*
2. *Y contains an edge which intersects $\hat{\mathfrak{h}}$.*
3. *Both $\mathfrak{h}$ and $\mathfrak{h}^*$ intersect Y.*

Proof. Assume that $\hat{\mathfrak{h}}$ passes through Y. Let C be a cube of Y, which intersects $\hat{\mathfrak{h}}$. Then $C \cap \hat{\mathfrak{h}}$ is a midcube of C. Let e be the edge whose midpoint is a vertex of $C \cap \hat{\mathfrak{h}}$. Then e intersects Y and $e \subset C \subset Y$.

Suppose that Y contains an edge e which intersects $\hat{\mathfrak{h}}$. Then the halfspace $\mathfrak{h}$ of $\hat{\mathfrak{h}}$ contains an endpoint of e. Then its opposite halfspace contains the other endpoint of e. So both $\mathfrak{h}$ and $\mathfrak{h}^*$ intersect Y.

Assume that both $\mathfrak{h}$ and $\mathfrak{h}^*$ intersect Y. Choose vertices $y_1 \in \mathfrak{h} \cap Y$ and $y_2 \in \mathfrak{h}^* \cap Y$. Since Y is connected, there is a path $\rho : [0,1] \to \tilde{X}$ joining y_1 to y_2. Because $\mathfrak{h}$ is disjoint from $\mathfrak{h}^*$, there is $t_0 \in [0,1]$ such that $\rho(t_0)$ is contained in neither $\mathfrak{h}$ nor $\mathfrak{h}^*$. Let C' be the cube such that $\rho(t_0)$ is in the interior of C'. Then $C' \subset Y$ and C' belongs to neither $\mathfrak{h}$ nor $\mathfrak{h}^*$. By the maximality of halfspaces, there exists an edge e of C', which is contained in neither $\mathfrak{h}$ nor $\mathfrak{h}^*$. So, by Remark 5.2.12, the edge e intersects $\hat{\mathfrak{h}}$. So $\hat{\mathfrak{h}}$ intersects Y. □

Definition 5.2.21. For a CAT(0) cube complex $\tilde{X}$, the set of all halfspaces in $\tilde{X}$ with the opposite operator is called the *combinatorial pocset* of $\tilde{X}$. And the set of all hyperplanes in $\tilde{X}$ is said to be the *combinatorial wall structure* of $\tilde{X}$.

Remark 5.2.22. The opposite operator has order 2. That is, $(\mathfrak{h}^*)^* = \mathfrak{h}$ for all halfspace $\mathfrak{h}$ in a CAT(0) cube complex.

Lemma 5.2.23. *Let $\tilde{X}$ be a* CAT(0) *cube complex. If $\mathfrak{h}_1$ is a halfspace in $\tilde{X}$ and a hyperplane $\hat{\mathfrak{h}}_2$ is contained in $\mathfrak{h}_1$, then some halfspace $\mathfrak{h}_2$ of $\hat{\mathfrak{h}}_2$ is properly contained in $\mathfrak{h}_1$, i.e., $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2$.*

Proof. By the maximality of halfspaces, every edge intersecting $\hat{\mathfrak{h}}_2$ is contained in $\mathfrak{h}_1$, and then $\mathcal{N}(\mathfrak{h}_2)$ is a subset of $\mathfrak{h}_1$. So every square in $\tilde{X}$, which intersects $\hat{\mathfrak{h}}_2$, does not intersect $\hat{\mathfrak{h}}_1$ (because it is contained in $\mathcal{N}(\hat{\mathfrak{h}}_2)$.) That is, $\hat{\mathfrak{h}}_1$ is not transverse to $\hat{\mathfrak{h}}_2$. By Proposition 5.2.15.(3), $\hat{\mathfrak{h}}_1$ is connected so that it is contained in only one of halfspaces of $\hat{\mathfrak{h}}_2$. Let $\mathfrak{h}_2$ be the halfspace which is disjoint from $\hat{\mathfrak{h}}_1$. Since $\mathfrak{h}_2$ is connected, it is also contained in only one of halfspaces of $\hat{\mathfrak{h}}_1$. On the other hand, because $\mathfrak{h}_2$ intersects the carrier of $\hat{\mathfrak{h}}_2$ and $\mathcal{N}(\hat{\mathfrak{h}}_2) \subset \mathfrak{h}_1$, the intersection of $\mathfrak{h}_2$ and $\mathfrak{h}_1$ is nonempty. So $\mathfrak{h}_2 \subseteq \mathfrak{h}_1$. Since $\mathcal{N}(\hat{\mathfrak{h}}_2) \subset \mathfrak{h}_1$, the halfspace $\mathfrak{h}_2$ cannot be equal to $\mathfrak{h}_1$. Hence, $\mathfrak{h}_2 \subsetneq \mathfrak{h}_1$. □

Lemma 5.2.24. *Let $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ be hyperplanes in a* CAT(0) *cube complex. If $\hat{\mathfrak{h}}_1$ osculates $\hat{\mathfrak{h}}_2$, then they do not cross each other and any hyperplane does not separate them.*

Proof. If $\hat{\mathfrak{h}}_1$ osculates $\hat{\mathfrak{h}}_2$, then they do not cross each other by Lemma 5.2.14. If some hyperplane $\hat{\mathfrak{h}}$ separates $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$, then $\hat{\mathfrak{h}}_1 \subset \mathfrak{h}$ and $\hat{\mathfrak{h}}_2 \subset \mathfrak{h}^*$ for some halfspace $\mathfrak{h}$ of $\hat{\mathfrak{h}}$. Then $\mathcal{N}(\hat{\mathfrak{h}}_1)$ is disjoint from $\mathcal{N}(\hat{\mathfrak{h}}_2)$. So $\hat{\mathfrak{h}}_1$ does not osculate $\hat{\mathfrak{h}}_2$. □

Remark 5.2.25. If $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are hyperplanes of a CAT(0) cube complex, then exactly one of the following is satisfied.

1. $\hat{\mathfrak{h}}_1$ is equal to $\hat{\mathfrak{h}}_2$.
2. $\hat{\mathfrak{h}}_1$ crosses $\hat{\mathfrak{h}}_2$.
3. $\hat{\mathfrak{h}}_1$ osculates $\hat{\mathfrak{h}}_2$.
4. Some hyperplane separates $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$.

Lemma 5.2.26. *Let $\Phi : \tilde{X} \to \tilde{Y}$ be a cubical isomorphism between two* CAT(0) *cube complexes.*

1. *An edge e of $\tilde{X}$ is parallel to e' if and only if $\Phi(e)$ is parallel to $\Phi(e')$.*
2. *A subset $\hat{\mathfrak{h}}$ of $\tilde{X}$ is a hyperplane if and only if $\Phi(\hat{\mathfrak{h}})$ is a hyperplane of $\tilde{Y}$.*
3. *A subcomplex $\mathfrak{h}$ of $\tilde{X}$ is a halfspace if and only if $\Phi(\mathfrak{h})$ is a halfspace of $\tilde{Y}$.*

Proof. (1) If e and e' are contained in a square S and they are parallel, then $\Phi(e)$ and $\Phi(e')$ are parallel to each other because Φ sends S to a square of $\tilde{Y}$ isometrically. In general, if e is parallel to e', then there are edges $e = e_1, e_2, \ldots, e_{n-1}, e_n = e'$ such that e_i and e_{i+1} are contained in a square S_i and they are parallel for each $i = 1, \ldots, n-1$. Since $\Phi(e_i)$ is parallel to $\Phi(e_{i+1})$ for each $i = 1, \ldots, n-1$, we have $\Phi(e)$ is parallel to $\Phi(e')$.

(2) Let $\hat{\mathfrak{h}}$ be a hyperplane of $\tilde{X}$. Let $[e]$ be the parallelism class such that $\hat{\mathfrak{h}}$ intersects edges of $[e]$. By (1), the image $[\Phi(e)]$ of $[e]$ is a parallelism class in $\tilde{Y}$. Then $\Phi(\hat{\mathfrak{h}})$ is the union of midcubes intersecting edges of $[\Phi(e)]$ so it is a hyperplane of $\tilde{Y}$.

(3) Let $\mathfrak{h}$ be a halfspace of $\tilde{X}$. If $\hat{\mathfrak{h}}$ is the hyperplane of $\mathfrak{h}$, then $\Phi(\hat{\mathfrak{h}})$ is a hyperplane in $\tilde{Y}$ by (2). Because $\Phi(\mathfrak{h})$ is connected and disjoint from $\Phi(\hat{\mathfrak{h}})$, some halfspace $\mathfrak{h}'$ of $\Phi(\hat{\mathfrak{h}})$ contains $\Phi(\mathfrak{h})$. Similarly, since $\Phi^{-1}(\mathfrak{h}')$ is connected and disjoint from $\hat{\mathfrak{h}}$, we have $\Phi^{-1}(\mathfrak{h}') \subset \mathfrak{h}$. Hence, $\Phi(\mathfrak{h}) = \mathfrak{h}'$, i.e., it is a halfspace of $\Phi(\hat{\mathfrak{h}})$. □

5.3 Orientations

Definition 5.3.1. Let $\tilde{X}$ be a CAT(0) cube complex. And let $\vec{e}$ be an oriented edge on $\tilde{X}$ and $\hat{\mathfrak{h}}$ the hyperplane of $\vec{e}$. The *terminal halfspace* of $\vec{e}$ is the halfspace of $\hat{\mathfrak{h}}$ containing the terminal vertex of $\vec{e}$. Similarly, the *initial halfspace* of $\vec{e}$ is the halfspace of $\hat{\mathfrak{h}}$ containing the initial vertex of $\vec{e}$. A parallelism class is *oriented* if all elements in the class are oriented and their termimal halfspaces are equal.

Definition 5.3.2. Let $\mathcal{H}$ be the combinatorial pocset of a CAT(0) cube complex $\tilde{X}$. We say that $\tilde{X}$ is *oriented* if all parallelism classes are oriented. An *orientation* $\mathcal{O}$ of $\tilde{X}$ is a subset of $\mathcal{H}$ satisfying that $\{\mathfrak{h}, \mathfrak{h}^*\} \cap \mathcal{O}$ has exactly one halfspace for all $\mathfrak{h} \in \mathcal{H}$.

Remark 5.3.3. Let $\tilde{X}$ be a CAT(0) cube complex. Choosing an orientation of $\tilde{X}$ is equivalent to giving orientations on all parallelism classes. Precisely, the following statements give an one-to-one correspondence.

- If an orientation $\mathcal{O}$ of $\tilde{X}$ is determined, then, for each parallelism class $[e]$, we give the orientation on $[e]$ satisfying that its terminal halfspace belongs to $\mathcal{O}$.
- Whenever all parllelism classes are oriented, the set of terminal halfspaces is an orientation of $\tilde{X}$.

The next definition is an example of orientations.

Definition 5.3.4. Let $\tilde{X}$ be a CAT(0) cube complex and x a vertex of $\tilde{X}$. The *outward orientation* about x, denoted by $\mathcal{O}(x)$, is the set of halfspaces which does not contains x. The *inward orientation* about x, denoted by $\mathcal{I}(x)$, is the set of halfspaces which contains x.

Remark 5.3.5. Let $\tilde{X}$ be a CAT(0) cube complex and $\mathcal{H}$ the combinatorial pocset of $\tilde{X}$.

1. For each vertex x of $\tilde{X}$, every halfspace is contained in either the inward orientation of x or the outward orientation of x.
2. If two vertices x and y are distinct, then there is a hyperplane $\hat{\mathfrak{h}}$ which separates x and y. Then some halfspace $\mathfrak{h}$ of $\hat{\mathfrak{h}}$ belongs to the inward orientation $\mathcal{I}(x)$ of x and its opposite halfspace $\mathfrak{h}^*$ is contained in the inward orientation $\mathcal{I}(y)$ of y. So $\mathcal{I}(x)$ is distinct from $\mathcal{I}(y)$.

5.4 Combinatorial metrics

Definition 5.4.1 (length metric)**.** Let (X, d) be a metric space and Y a subset of X. Let $I \subset \mathbb{R}$ be a compact interval The *length* of a (continuous) path $\phi : I \to X$ in X, denoted by $l(\phi)$, is

$$l(\phi) := \sup \left\{ \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \,|\, \{t_1 < \cdots < t_n\} \subset I,\ x_i = \phi(t_i) \right\}.$$

Then the *length metric* d' on Y, induced from d, is the metric on Y defined by

$$d'(y_1, y_2) := \inf\{l(\phi) \,|\, \phi : [0, 1] \to Y \text{ is a path joining } y_1 \text{ and } y_2\}$$

for all $y_1, y_2 \in Y$.

Definition 5.4.2. The *combinatorial metric* of a CAT(0) cube complex $\tilde{X}$ is the length metric on the 1-skeleton $\tilde{X}^{(1)}$, induced from the metric of $\tilde{X}$.

Remark 5.4.3. If d is the combinatorial metric of a CAT(0) cube complex $\tilde{X}$, then we often give the subspace metric of d on the 0-skeleton $\tilde{X}^{(0)}$.

Definition 5.4.4. Let $\tilde{X}$ be a CAT(0) cube complex. Let A be either $\{1, \ldots, n\}$, $\mathbb{N}$ or $\mathbb{Z}$. A *combinatorial path* is a sequence $(x_i)_{i\in A}$ of vertices of $\tilde{X}$ such that x_i and x_{i+1} are joined by an edge for all $i \in A$. Equivalently, a *combinatorial path* is a sequence $(\vec{e}_i)_{i\in A}$ of oriented edges such that the terminal vertex of $\vec{e}_i$ is the initial vertex of $\vec{e}_{i+1}$ for all $i \in A$.

Definition 5.4.5. The *length* of a combinatorial path is the number of edges in the path. If a combinatorial path has infinitely many edges, then we define its length as ∞.

Definition 5.4.6. A combinatorial path $\vec{L}$ on a CAT(0) cube complex is called a *(combinatorial) geodesic path* if, whenever a subpath $\vec{L}'$ of $\vec{L}$ has finite length, the length of $\vec{L}'$ is equal to the combinatorial metric of its endpoints.

Definition 5.4.7. Let $\tilde{X}$ be a CAT(0) cube complex. A connected 1-dimensional subcomplex L of $\tilde{X}$ is a *combinatorial geodesic* if L is a geodesic with respect to the combinatorial metric of $\tilde{X}$.

Proposition 5.4.8 (Lemma 13.1 in [21], [35], [20]). *Let $\tilde{X}$ be a* CAT(0) *cube complex with the combinatorial metric d. Let x and y be vertices in $\tilde{X}$. Then $d(x, y)$ is the number of hyperplanes seperating x and y. In general, a combinatorial path $\vec{L} = (\vec{e}_i)_{i \in A}$ ($A = \{1, \dots, n\}$, $\mathbb{N}$ or $\mathbb{Z}$) is a geodesic path if and only if $\vec{e}_i$ is not parallel to $\vec{e}_j$, as unoriented edges, for all distinct $i, j \in A$.*

Remark 5.4.9. Let x and y be vertices of a CAT(0) cube complex and $\mathcal{H}$ the combinatorial pocset. If d is the combinatorial metric and $\mathcal{I}(x)$ and $\mathcal{I}(y)$ are the inward orientations of x and y, then

$$\begin{aligned} d(x, y) &= |\{\mathfrak{h} \in \mathcal{H} \mid x \in \mathfrak{h},\ y \in \mathfrak{h}^*\}| \\ &= |\mathcal{I}(x) - \mathcal{I}(y)|. \end{aligned}$$

Similarly, if $\mathcal{O}(x)$ and $\mathcal{O}(y)$ are the outward orientations of x and y, respectively, then $d(x, y) = |\mathcal{O}(x) - \mathcal{O}(y)|$.

Lemma 5.4.10 (Lemma 2.2 in [12]). *Let $\tilde{X}$ be a* CAT(0) *cube complex with the combinatorial metric d. Let $\vec{L} = (\vec{e}_i)_{i \in A}$ ($A = \{1, \dots, n\}$, $\mathbb{N}$ or $\mathbb{Z}$) be a combinatorial path. For each $i \in A$, let $\mathfrak{h}_i$ be the terminal halfspace of e_i. Then the following statements are equivalent.*

1. *$\vec{L}$ is a geodesic path.*
2. *If $i < j$, then either $\mathfrak{h}_i \supsetneq \mathfrak{h}_j$ or $\mathfrak{h}_i$ is tansverse to $\mathfrak{h}_j$.*

Proof. ((1)⇒(2)) Assume that $\vec{L}$ is a geodesic path. Choose $i, j \in A$ with $i < j$. Then $\vec{e}_i$ is not parallel to $\vec{e}_j$ by Proposition 5.4.8. So $\mathfrak{h}_i$ is neither $\mathfrak{h}_j$ nor $\mathfrak{h}_j^*$.

Again, by Proposition 5.4.8, $\vec{e}_j$ is not parallel to any edge of $\vec{L} \setminus \{\vec{e}_j\}$. It implies that $\vec{e}_l$ cannot connect $\mathfrak{h}^*$ and $\mathfrak{h}$ for any $l < j$. Since the initial vertex of $\vec{e}_j$ is contained in $\mathfrak{h}_j^*$, we have $\vec{e}_i \in \mathfrak{h}_j^*$. Then $\mathfrak{h}_j^*$ intersects both $\mathfrak{h}_i$ and $\mathfrak{h}_i^*$. So $\mathfrak{h}_i$ cannot contain $\mathfrak{h}_j^*$. Hence, either $\mathfrak{h}_i \supsetneq \mathfrak{h}_j$ or they are transverse to each other.

((2)⇒(1)) Assume the statement (2) holds for $\vec{L}$. Then $\mathfrak{h}_i \neq \mathfrak{h}_j$ for all distinct $i, j \in A$. So $\vec{L}$ is a geodesic path by Proposition 5.4.8. □

Definition 5.4.11. A subcomplex Y of a CAT(0) cube complex $\tilde{X}$ is called *convex* if it is convex with respect to the CAT(0) metric of $\tilde{X}$.

By definition, a convex subcomplex of a CAT(0) cube complex is a CAT(0) cube complex itself.

Lemma 5.4.12 (Proposition 13.7 in [21]). *Let $\tilde{X}$ be a* CAT(0) *cube complex.*

1. *Let Y be a convex subcomplex of $\tilde{X}$. If x and y are vertices on Y and L is a combinatorial geodesic of $\tilde{X}$ joining x and y, then L is contained in Y.*

2. *Suppose that a subcomplex Y of $\tilde{X}$ satisfies the following: whenever x and y are vertices of Y and L is a combinatorial geodesic of $\tilde{X}$ joining x and y, we have $L \subset Y$. Then there is a convex subcomplex whose* 1*-skeleton is equal to $Y^{(1)}$.*

Lemma 5.4.13 (Lemma 13.4 in [21]). *In a* CAT(0) *cube complex, all halfspaces and the carriers of all hyperplanes are convex.*

Remark 5.4.14. By Lemma 5.4.13, we have $\mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}$ and $\mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}^*$ are convex for every halfspace $\mathfrak{h}$.

Proposition 5.4.15 (Lemma 13.6 in [21]). *Let $\tilde{X}$ be a* CAT(0) *cube complex and $\mathcal{H}$ the set of all halfspaces of $\tilde{X}$.*

1. *For every subset $\mathcal{H}'$ of $\mathcal{H}$, the intersection of all halfspaces in $\mathcal{H}'$ is convex or empty.*

2. *If Y is a convex subcomplex of $\tilde{X}$ and $\mathcal{I}(Y) = \{\mathfrak{h} \in \mathcal{H} \,|\, Y \subset \mathfrak{h}\}$, then $Y = \bigcap_{\mathfrak{h} \in \mathcal{I}(Y)} \mathfrak{h}$.*

Remark 5.4.16. Let $\tilde{X}$ be a CAT(0) cube complex and Y a convex subcomplex of $\tilde{X}$. Let $d^{\mathbb{E}}_{\tilde{X}}$ be the CAT(0) metric of $\tilde{X}$. If $d^{\mathbb{E}}_{Y}$ is the metric on Y induced by $d^{\mathbb{E}}_{\tilde{X}}$, then $(Y, d^{\mathbb{E}}_{Y})$ is a CAT(0) space so that Y is a CAT(0) cube complex. But the converse does not hold. For example, if $X = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ is the cube complex such that all integer pairs are vertices of X. Then X with the Euclidean metric is a CAT(0) cube complex. Let X' be a subcomplex which contains exactly three squares of X. If X' does not contains the vertex $(1, 1) \in X$, then the geodesic joining $(1, 0)$ and $(0, 1)$ in X does not lie in X'. So X' is not a convex subcomplex of X. Nevertheless, the link of every vertex of X' is flag so that X' is a CAT(0) cube complex itself.

Lemma 5.4.17 (Lemma 13.13 in [21])**.** *If $Y_1, \ldots, Y_n$ are convex subcomplexes of a* CAT(0) *cube complex and they intersect pairwise, then $\bigcap_{i=1}^{n} Y_i$ is nonempty.*

Lemma 5.4.18. *A convex subcomplex C of a* CAT(0) *cube complex $\tilde{X}$ is a cube if and only if all hyperplanes, passing through C, cross each other.*

Proof. If C is a cube, then it is obvious that all hyperplanes intersecting C are pairwise transverse. If C is not a cube, then there are two edges e_1 and e_2 in C such that they share a vertex but do not belong to the same square. Then their hyperplanes osculate each other so that they do not cross by Lemma 5.2.24. □

Lemma 5.4.19 (Corollary 3.14, [41])**.** *If $\mathfrak{h}$ is a halfspace of a* CAT(0) *cube complex, then $\mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$ is convex.*

Proof. Choose two vertices x and y in $\mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$. If x and y belong to $\mathfrak{h}$, then every combinatorial geodesic joining x and y lies in $\mathfrak{h} \subseteq \mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$ since $\mathfrak{h}$ is convex. If x and y are contained in $\mathcal{N}(\hat{\mathfrak{h}})$, then every combinatorial geodesic joining x and y lies in $\mathcal{N}(\hat{\mathfrak{h}}) \subseteq \mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$ since $\mathcal{N}(\hat{\mathfrak{h}})$ is convex. Assume that $x \in \mathcal{N}(\hat{\mathfrak{h}}) \cap H^*$ and $y \in \mathfrak{h}$. If $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ is a combinatorial geodesic joining x and y, then there is some $j \in \{1, \ldots, n\}$ such that the edge connecting x_{j-1} and x_j is dual to $\hat{\mathfrak{h}}$. Since $x_{j-1} \in \mathcal{N}(\hat{\mathfrak{h}})$ and the sequence $x_0, \ldots, x_{j-1}$ is a combinatorial geodesic joining x_0 and x_{j-1}, all vertices of $x_0, \ldots, x_{j-1}$ lie in $\mathcal{N}(\hat{\mathfrak{h}})$ by convexity. Similarly, all vertices of $x_j, \ldots, x_n$ are in $\mathfrak{h}$. So the geodesic $x_0, \ldots, x_n$ is contained in $\mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$. Hence, $\mathfrak{h} \cup \mathcal{N}(\hat{\mathfrak{h}})$ is convex. □

Definition 5.4.20. Let Y be a convex subcomplex of a CAT(0) cube complex $\tilde{X}$ and x a vertex of $\tilde{X}$. Let d be the combinatorial metric on $\tilde{X}$. Then the *combinatorial distance* $d(x, Y)$ between x and Y is defined by the minimum value among $d(x, y)$ for all vertices y in Y.

Lemma 5.4.21 (Lemma 13.8 in [21])**.** *Let $\tilde{X}$ be a* CAT(0) *cube complex with the combinatorial metric d. Let Y be a convex subcomplex of $\tilde{X}$. Then there is a cubical map $\pi_Y : \tilde{X} \to \tilde{X}$ such that*

1. *$\pi_Y(\tilde{X}) = Y$,*
2. *$d(x, y) = d(x, \pi_Y(x)) + d(\pi_Y(x), y)$ for all $x \in \tilde{X}$ and $y \in Y$, and,*

3. *for each vertex x of $\tilde{X}$, a hyperplane $\hat{\mathfrak{h}}$ separates x and $\pi_Y(x)$ if and only if $\hat{\mathfrak{h}}$ separates x and Y.*

As a corollary, $d(x, \pi_Y(x)) = d(x, Y)$.

Definition 5.4.22. The cubical map defined in Lemma 5.4.21 is called the *closest-point projection* from $\tilde{X}$ to Y.

Lemma 5.4.23. *Let Y and Z be disjoint convex subcomplexes of a* CAT(0) *cube complex. Assume that two vertices $y \in Y$ and $z \in Z$ satisfy that $d(y, z) = d(Y, Z)$. Then, a hyperplane $\hat{\mathfrak{h}}$ separates y and z if and only if $\hat{\mathfrak{h}}$ separates Y and Z.*

Proof. Since z is closest to y among vertices of Z, we have $z = \pi_Z(y)$ by Lemma 5.4.21. Similarly, $y = \pi_Y(z)$. If $\hat{\mathfrak{h}}$ separates y from z, then $\hat{\mathfrak{h}}$ separates y from Z and also separates z from Y by Lemma 5.4.21.(3). Hence, $\hat{\mathfrak{h}}$ separates Y from Z. □

5.5 Dual cube complexes of finite-width discrete pocsets

Definition 5.5.1. A *discrete pocset* is a partially ordered set $(\mathcal{P}, \prec)$ with an *opposite operator* $* : \mathcal{P} \to \mathcal{P}$ satisfying that

1. $P^* \neq P$ for all $P \in \mathcal{P}$,
2. $(P^*)^* = P$ for all $P \in \mathcal{P}$,
3. $P^* \succ Q^*$ whenever $P \prec Q$, and
4. the set $\{R \in \mathcal{P} \mid P \prec R \prec Q\}$ is finite whenever $P \prec Q$.

Remark 5.5.2. In general, a *pocset* is a partially ordered set with an opposite operator satisfying (1), (2) and (3) of Definition 5.5.1. But we will not deal with any non-discrete pocset.

Definition 5.5.3. Let $(\mathcal{P}, \prec)$ be a discrete pocset. A subset $\mathcal{Q}$ of $\mathcal{P}$ is called an *ultrafilter* if

1. $\{P, P^*\} \cap \mathcal{Q}$ has exactly one element for all $P \in \mathcal{P}$ and,

2. whenever $P \in \mathcal{Q}$ and $P \prec Q$, we have $Q \in \mathcal{Q}$.

An ultrafilter $\mathcal{Q}$ of $\mathcal{P}$ is said to satisfy the *descending chain condition* or is called a *principal ultrafilter* if every descending chain of $\mathcal{Q}$ terminates.

Definition 5.5.4. A pair of elements P and Q in a discrete pocset $\mathcal{P}$ are said to be *transverse* to each other if $P \not\prec Q$, $P \not\succ Q$, $P \not\prec Q^*$ and $P \not\succ Q^*$. A discrete pocset is called *finite-width* if the number of pairwise transverse elements is uniformly bounded.

Remark 5.5.5. For a CAT(0) cube complex $\tilde{X}$, the combinatorial pocset $\mathcal{H}$ with the set-inclusion and the opposite operator forms a discrete pocset. For each vertex $x \in \tilde{X}^{(0)}$, the inward orientation $\mathcal{I}(x)$ of x is a principal ultrafilter. The converse is also true. If $\mathcal{Q}$ is a principal ultrafilter of $\mathcal{H}_{\tilde{X}}$, then $\bigcap_{\mathfrak{h}\in\mathcal{Q}} \mathfrak{h}$ is a vertex. So $\mathcal{Q}$ is the inward orientation of $\bigcap_{\mathfrak{h}\in\mathcal{Q}} \mathfrak{h}$.

If $\tilde{X}$ is n-dimensional, then the number of pairwise transverse halfspaces are at most n. So $\mathcal{H}$ is finite-width.

Definition 5.5.6. Two discrete pocsets $\mathcal{P}$ and $\mathcal{Q}$ are *isomorphic* if there is an order-preserving bijective map $\mathcal{P} \to \mathcal{Q}$ which also perserves opposite operators.

Remark 5.5.7. Let $\mathcal{P}$ and $\mathcal{Q}$ be discrete pocsets and $\phi : \mathcal{P} \to \mathcal{Q}$ a pocset isomorphism. Then the following can be obtained immediately.

1. For $P_1, P_2 \in \mathcal{P}$, if P_1 is transverse to P_2, then $\phi(P_1)$ is transverse to $\phi(P_2)$.

2. If $\mathcal{P}$ is finite-width, then $\mathcal{Q}$ is also finite-width.

3. If $\mathcal{U}$ is an ultrafilter of $\mathcal{P}$, then $\phi(\mathcal{U})$ is an ultrafilter.

4. If an ultrafilter $\mathcal{U}$ of $\mathcal{P}$ is principal, then $\phi(\mathcal{U})$ is also principal.

Proposition 5.5.8. *Let $\tilde{X}$ and $\tilde{Y}$ be* CAT(0) *cube complexes. Let $\mathcal{H}(\tilde{X})$ and $\mathcal{H}(\tilde{Y})$ be the combinatorial pocsets of $\tilde{X}$ and $\tilde{Y}$, respectively. Then, $\tilde{X}$ is isomorphic to $\tilde{Y}$ if and only if $\mathcal{H}(\tilde{X})$ is isomorphic to $\mathcal{H}(\tilde{Y})$.*

Proof. Assume that there is a cubical isomorphism $\Phi : \tilde{X} \to \tilde{Y}$. By Lemma 5.2.26.(3), the map $\phi : \mathcal{H}(\tilde{X}) \to \mathcal{H}(\tilde{Y})$, defined by $\phi(\mathfrak{h}) := \Phi(\mathfrak{h})$ for all $\mathfrak{h} \in \mathcal{H}(\tilde{X})$, is well-defined and bijective. Set-theoretically, ϕ preserves partial orders and opposite operators. So ϕ is a pocset isomorphism between $\mathcal{H}(\tilde{X})$ and $\mathcal{H}(\tilde{Y})$.

Conversely, assume that there is a pocset isomorphism $\phi : \mathcal{H}(\tilde{X}) \to \mathcal{H}(\tilde{Y})$. For each vertex x of $\tilde{X}$, let $\mathcal{I}(x)$ be the inward orientation of x. First, for each vertex x of $\tilde{X}$, define

$$\Phi_0(x) := \bigcap_{\mathfrak{h} \in \mathcal{I}(x)} \phi(\mathfrak{h}).$$

Since $\mathcal{I}(x)$ is a principal ultrafilter (by Remark 5.5.5), $\phi(\mathcal{I}(x))$ is a principal ultrafilter by Remark 5.5.7. So $\Phi_0(x)$ is a vertex of $\tilde{Y}$ by Remark 5.5.5. Therefore, $\Phi_0 : \tilde{X}^{(0)} \to \tilde{Y}^{(0)}$ is well-defined. If we define $\overline{\Phi}_0 : \tilde{Y}^{(0)} \to \tilde{X}^{(0)}$ by $\overline{\Phi}_0(y) = \bigcap_{\mathfrak{h} \in \mathcal{I}(y)} \phi^{-1}(\mathfrak{h})$ for all $y \in \tilde{Y}^{(0)}$, then $\overline{\Phi}_0$ is also well-defined and is the inverse of Φ_0. So Φ_0 is bijective.

Second, we will define a map $\Phi : \tilde{X} \to \tilde{Y}$ satisfying that the restriction of Φ on the 0-skeleton of $\tilde{X}$ is equal to Φ_0. Let C be an n-cube of $\tilde{X}$ and $\{\hat{\mathfrak{h}}_1, \ldots, \hat{\mathfrak{h}}_n\}$ the set of hyperplanes intersecting C. Let $\mathfrak{h}_i$ be a halfspace of $\hat{\mathfrak{h}}_i$ for each $i = 1, \ldots, n$. Then, for distinct $i, j \in \{1, \ldots, n\}$, we have $\mathfrak{h}_i$ is transverse to both $\mathfrak{h}_j$ and $\mathfrak{h}_j^*$. If $\mathcal{I}(C)$ is the set of halfspaces containing C and $\mathcal{O}(C) = \{\mathfrak{h}^* \mid \mathfrak{h} \in \mathcal{H}(C)\}$, then $C = \bigcap_{\mathfrak{h} \in \mathcal{I}(C)} \mathfrak{h}$ and

$$\mathcal{H}(\tilde{X}) = \mathcal{I}(C) \sqcup \mathcal{O}(C) \sqcup \{\mathfrak{h}_1, \ldots, \mathfrak{h}_n, \mathfrak{h}_1^*, \ldots, \mathfrak{h}_n^*\}.$$

Let $C' = \bigcap_{\mathfrak{h} \in \mathcal{I}(C)} \phi(\mathfrak{h})$. Note that, for each vertex $x \in C$, the inward orientation of x contains $\mathcal{I}(C)$ as a subset. Then $\Phi_0(x) = \bigcap_{\mathfrak{h} \in \mathcal{I}(x)} \phi(\mathfrak{h}) \in C'$ for all vertices $x \in C$. So C' is a nonempty convex subcomplex by Proposition 5.4.15. For each $i = 1, \ldots, n$, let $\hat{\mathfrak{h}}_i'$ be the hyperplane whose halfspaces are $\phi(\mathfrak{h}_i)$ and $\phi(\mathfrak{h}_i^*)$. Since both $\phi(\mathfrak{h}_i)$ and $\phi(\mathfrak{h}_i^*)$ intersect C', the hyperplane $\hat{\mathfrak{h}}_i'$ passes through C' for all i. Other hyperplanes of $\tilde{Y}$ do not intersect C' because their halfspaces are contained in $\phi(\mathcal{I}(C)) \cup \phi(\mathcal{O}(C))$. Because $\hat{\mathfrak{h}}_1', \ldots, \hat{\mathfrak{h}}_n'$ are pairwise transverse, C' is an n-cube by Lemma 5.4.18. In conclusion, for every n-cube C of $\tilde{X}$, there is a unique n-cube $\Phi(C)$ in $\tilde{Y}$ and an isometry $\Phi_C : C \to \Phi(C)$ such that $\Phi_C(x) = \Phi_0(x)$ for every vertex $x \in C$. Define $\Phi : \tilde{X} \to \tilde{Y}$ satisfying that, if $x \in C \subset \tilde{X}$, then $\Phi(x) = \Phi_C(x)$.

Third, we will show that Φ is a well-defined. It is enough to prove that, for a subcube F of a cube C in $\tilde{X}$, the restriction of Φ_C to F is equal to Φ_F. For each vertex x in F, we have $\Phi_C(x) = \Phi_0(x) = \Phi_F(x)$. Since Φ_F and Φ_C are determined by Φ_0, we have $\Phi_F(x) = \Phi_C(x)$ for all $x \in F$.

Finally, we claim that Φ is a cubical isomorphism. We need only to show that Φ is an isometry with respect to CAT(0) metrics. For each cube C' of $\tilde{Y}$, the intersection $\bigcap_{\mathfrak{h}' \supset C'} \phi^{-1}(\mathfrak{h}')$ is a cube and its image under Φ is exactly C'. So Φ is bijective. Note that Φ and Φ^{-1} send paths to paths with the same lengths. Hence, Φ is an isometry. $\square$

Theorem 5.5.9 (Theorem 10.3 in [33], [35], [36])**.** *For every finite-width discrete pocset* $(\mathcal{P}, \prec)$*, there is a finite-dimensional* CAT(0) *cube complex* $\tilde{X}$ *such that the combinatorial pocset of* $\tilde{X}$ *is isomorphic to* $\mathcal{P}$*.*

Remark 5.5.10. Let $\mathcal{P}$ be a finite-width discrete pocset and $\tilde{X}$ a dual cube complex of $\mathcal{P}$. If $\tilde{X}'$ is another CAT(0) cube complex obtained from $\mathcal{P}$ by Theorem 5.5.9, then it is isomorphic to $\tilde{X}$ by Proposition 5.5.8.

Definition 5.5.11. Let $\mathcal{P}$ be a finite-width discrete pocset. Then, by Theorem 5.5.9, there is a CAT(0) cube complex $\tilde{X}$ such that the set $\mathcal{H}$ of halfspaces in $\tilde{X}$ is isomorphic to $\mathcal{P}$. We call $\tilde{X}$, with a pocset-isomorphism $\mathcal{P} \to \mathcal{H}$, the *dual cube complex* of $\mathcal{P}$.

Example 2 (Wallspaces)**.**

1. ([20], [41]) Let $\mathcal{X}$ be a set. A *wall partition* of $\mathcal{X}$ is a pair of two subsets H_1 and $\mathsf{H}_2\}$ such that $\mathsf{H}_1 \cup \mathsf{H}_2 = \mathcal{X}$ and we call each component of a wall partition a *halfspace*. (It is not necessary that H_1 is disjoint from H_2.) Let $\mathcal{W}$ be a set of wall partitions of $\mathcal{X}$ and $\mathcal{H}(\mathcal{W})$ the set of halfspaces in wall partitions of $\mathcal{W}$. Define a map $* : \mathcal{H} \to \mathcal{H}$ by $\mathsf{H} \cup \mathsf{H}^*$ satisfying that the image of every halfspace H, denoted by H^* forms a wall partition with H. Then $(\mathcal{H}, \subset, *)$ is a pocset. We call the pair $(\mathcal{X}, \mathcal{W})$ a *wallspace* or a *space with walls* if $\mathcal{H}(\mathcal{W})$ is a finite-width discrete pocset. Every wallspace has its dual cube complex by Theorem 5.5.9.

2. Let $\mathcal{X}$ be a topological space. In many cases, both halfspaces H and H^* in a wall partition are closed and $\mathsf{H} \cap \mathsf{H}^*$ is nonempty. We say that $\tilde{\gamma} := \mathsf{H} \cap \mathsf{H}^*$ is a *wall* if it is nonempty.

3. If $\tilde{\gamma}$ is a geodesic in the hyperbolic plane $\mathbb{H}^2$, then $\mathbb{H}^2 \setminus \tilde{\gamma}$ has two connected componenets. We call the closure of each component a *halfspace* bounded by $\tilde{\gamma}$. If H denotes a halfspace bounded by $\tilde{\gamma}$, then H and $\mathsf{H}^* := \overline{\mathbb{H}^2 \setminus \mathsf{H}}$ form a wall partition of $\mathbb{H}^2$ with the wall $\tilde{\gamma}$.

4. ([35], [36]) Let Σ be a hyperbolic surface of finite area and $\tilde{\gamma}$ a closed geodesic of Σ. Let $\mathcal{L}(\tilde{\gamma})$ be the set of all geodesics in $\mathbb{H}^2$ whose image

under the hyperbolic structure $\mathbb{H}^2 \to \Sigma$ is $\tilde{\gamma}$. Then $(\mathbb{H}^2, \mathcal{L}(\tilde{\gamma}))$ is a wallspace.

Example 3. Consider the 0-skeleton $\tilde{X}^{(0)}$ of a finite-dimensional CAT(0) cube complex $\tilde{X}$. For each hyperplane $\hat{\mathfrak{h}}$ of $\tilde{X}$, we obtain the decomposition $\tilde{X}^{(0)} = \mathfrak{h}^{(0)} \sqcup \mathfrak{h}^{*(0)}$, which is a wall partition of $\tilde{X}^{(0)}$. So $(\tilde{X}^{(0)}, \hat{\mathcal{H}})$ is a wallspace. Then its dual cube complex is isomorphic to $\tilde{X}$ by Proposition 5.5.8.

5.6 Isomorphisms and isometries

For a CAT(0) cube complex $\tilde{X}$, we can consider two kinds of symmetries, which are cubical isomorphisms and isometries preserving the combinatorial metric on $\tilde{X}$. We will show that they are not different essentially by a sequence of lemmas.

Lemma 5.6.1. *Let $(\tilde{X}, d_{\tilde{X}})$ and $(\tilde{Y}, d_{\tilde{Y}})$ be* CAT(0) *cube complexes with the combinatorial metrics. If $\Phi : \tilde{X} \to \tilde{Y}$ is a cubical isomorphism, then $\Phi_0 : \tilde{X}^{(0)} \to \tilde{Y}^{(0)}$, defined by $\Phi_0(x) = \Phi(x)$ for all $x \in \tilde{X}^{(0)}$, is an isometry between two combinatorial metrics.*

Proof. Choose vertices $x_1, x_2 \in \tilde{X}$. By Lemma 5.2.26.(3), the inward orientation $\mathcal{I}(\Phi(x_1))$ is equal to $\{\Phi(\mathfrak{h}) \,|\, \mathfrak{h} \in \mathcal{I}(x_1)\}$. Then, by Remark 5.4.9,

$$\begin{aligned} d_{\tilde{Y}}(\Phi(x_1), \Phi(x_2)) &= |\mathcal{I}(\Phi(x_1)) \setminus \mathcal{I}(\Phi(x_2))| \\ &= |\{\Phi(\mathfrak{h}) \,|\, \mathfrak{h} \in \mathcal{I}(x_1) \setminus \mathcal{I}(x_2)\}| \\ &= |\mathcal{I}(x_1) \setminus \mathcal{I}(x_2)| \\ &= d_{\tilde{X}}(x_1, x_2). \end{aligned}$$

So the restriction of Φ to the 0-skeletons is an isometry with respect to their combinatorial metrics. □

Proposition 5.6.2. *Let $\tilde{X}$ and $\tilde{Y}$ be* CAT(0) *cube complexes. Let $d_{\tilde{X}}$ and $d_{\tilde{Y}}$ be the combinatorial metrics on the 0-skeletons $\tilde{X}^{(0)}$ and $\tilde{Y}^{(0)}$, respectively. Then the following are equivalent.*

1. *$\tilde{X}$ is isomorphic to $\tilde{Y}$.*
2. *$(\tilde{X}^{(0)}, d_{\tilde{X}})$ is isometric to $(\tilde{Y}^{(0)}, d_{\tilde{Y}})$.*

Proof. Suppose that $\Phi_0 : \tilde{X}^{(0)} \to \tilde{Y}^{(0)}$ is an isometry with respect to their combinatorial metrics. If vertices x_1 and x_2 of $\tilde{X}$ are joined by an edge, then $d_{\tilde{Y}}(\Phi_0(x_1), \Phi_0(x_2)) = d_{\tilde{X}}(x_1, x_2) = 1$. So $\Phi_0(x_1)$ and $\Phi_0(x_2)$ are connected by an edge. It implies that there is a bijective cubical map $\Phi_1 : \tilde{X}^{(1)} \to \tilde{Y}^{(1)}$ which is an extension of Φ_0.

Let $x_0, \ldots, x_3$ be vertices of $\tilde{X}$, which form a square. We claim that there is a square whose vertices are $\Phi_0(x_0), \ldots, \Phi_0(x_3)$. Assume that

$$d_{\tilde{X}}(x_i, x_{i+1}) = 1$$

for each i (modulo 4). Let y_i denote $\Phi_0(x_i)$ and e'_i the edge joining y_i and y_{i+1}. Since Φ_0 is an isometry,

$$\begin{aligned} d_{\tilde{Y}}(y_0, y_2) = 2 = d_{\tilde{Y}}(y_0, y_1) + d_{\tilde{Y}}(y_1, y_2) \text{ and} \\ d_{\tilde{Y}}(y_0, y_2) = 2 = d_{\tilde{Y}}(y_0, y_3) + d_{\tilde{Y}}(y_3, y_2). \end{aligned}$$

So $e'_0 \cup e'_1$ and $e'_2 \cup e'_3$ are combinatorial geodesics joining y_0 and y_2. If a hyperplane $\hat{\mathfrak{h}}_0$ of $\tilde{Y}$ intersects e'_0, then it separates y_0 from y_2 so that it intersects either e'_2 or e'_3. Because e'_3 intersects e'_0 at y_0, we have e'_0 is parallel to e'_2. Similarly, e'_1 is parallel to e'_3. Note that $\hat{\mathfrak{h}}_0$ separates e'_1 from e'_3. So, if $\hat{\mathfrak{h}}_1$ is the hyperplane intersecting e'_1, then $\hat{\mathfrak{h}}_1$ intersects $\hat{\mathfrak{h}}_0$ transversally. If there is no square which is bounded by $e'_0, \ldots, e'_3$, then $\hat{\mathfrak{h}}_0$ also osculates $\hat{\mathfrak{h}}_1$, which is a contradiction by Lemma 5.2.24. So there is a square whose 1-skeleton is $e'_0 \cup e'_1 \cup e'_2 \cup e'_3$.

By the above, the image of each parallelism class under Φ_1 is a parallelism class of $\tilde{Y}$. So there is the induced map $\hat{\varphi}$ which sends each hyperplane of $\tilde{X}$ to a hyperplane of $\tilde{Y}$. Then $\hat{\varphi}$ preserves transverality. Let $\varphi : \mathcal{H}(\tilde{X}) \to \mathcal{H}(\tilde{Y})$ be the map such that $\varphi(\mathfrak{h})$ is the halfspace of the hyperplane $\hat{\varphi}(\hat{\mathfrak{h}})$ which contains vertices of $\Phi_0(\mathfrak{h}^{(0)})$ for each halfspace $\mathfrak{h}$ of a hyperplane $\hat{\mathfrak{h}}$. Then φ is a pocset isomorphism. So, by Proposition 5.5.8, there is an isomorphism between $\tilde{X}$ and $\tilde{Y}$. $\square$

Chapter 6

Subpocsets, collapsings and bridges

Recall that a pocset is a partially ordered set with an opposite operator. In this chapter, we focus on a pocset induced from a finite-dimensional CAT(0) cube complex to look into a convex subcomplex, a closest-point projection and a bridge.

6.1 Subpocsets

Definition 6.1.1. A *subpocset* $\mathcal{Q}$ of a pocset $(\mathcal{P}, \prec, *)$ is a subset of $\mathcal{P}$ which is a pocset itself with the partial order induced from $\mathcal{P}$ and the restriction of the opposite operator $*$ to $\mathcal{Q}$.

Proposition 6.1.2 ([10], [36]). *Let $\mathcal{Q}$ be a subpocset of a finite-width discrete pocset $\mathcal{P}$. Let $\tilde{X}(\mathcal{P})$ and $\tilde{X}(\mathcal{Q})$ be the dual cube complexes of $\mathcal{P}$ and $\mathcal{Q}$, respectively.*[1] *Let $\mathcal{H}(\mathcal{P})$ and $\mathcal{H}(\mathcal{Q})$ be the combinatorial pocsets of $\tilde{X}(\mathcal{P})$ and $\tilde{X}(\mathcal{Q})$, and $\tau : \mathcal{P} \to \mathcal{H}(\mathcal{P})$ and $\sigma : \mathcal{Q} \to \mathcal{H}(\mathcal{Q})$ pocset-isomorphisms. Then there is a surjective cubical map $\pi_{\mathcal{Q}} : \tilde{X}(\mathcal{P}) \to \tilde{X}(\mathcal{Q})$ such that, for every vertex $x \in \tilde{X}$, the inward orientation $\mathcal{I}(\pi_{\mathcal{Q}}(x))$ is equal to $\sigma(\tau^{-1}(\mathcal{I}(x)) \cap \mathcal{Q}) = \{\sigma(Q) \mid Q \in \mathcal{Q}, \tau(Q) \in \mathcal{I}(x)\}$.*

Definition 6.1.3. We call the surjective cubical map $\pi_{\mathcal{Q}}$ in Proposition 6.1.2 the *collapsing* from $\tilde{X}(\mathcal{P})$ to $\tilde{X}(\mathcal{Q})$.

[1] See Definition 5.5.11.

Remark 6.1.4. Let $(\mathcal{X}, \mathcal{W})$ be a wallspace. (The definition of wallspace is in Example 2.) Let $\mathcal{H}(\mathcal{W})$ be the collection of halfspaces of wall partitions in $\mathcal{W}$. For every subset $\mathcal{W}'$ of $\mathcal{W}$, the set $\mathcal{H}(\mathcal{W}') := \{\mathfrak{h} \,|\, \mathfrak{h}$ is a halfspace of a wall partition in $\mathcal{W}'\}$ is a subpocset of the set $\mathcal{H}(\mathcal{W})$. If $\tilde{X}(\mathcal{W})$ and $\tilde{X}(\mathcal{W}')$ are the dual cube complex of $\mathcal{H}(\mathcal{W})$ and $\mathcal{H}(\mathcal{W}')$, respectively, then by Proposition 6.1.2, the collapsing $\tilde{X}(\mathcal{W}) \to \tilde{X}(\mathcal{W}')$ always exists.

We will look at important examples of subpocsets and collapsings in subsections.

6.2 Convex subcomplexes

Remark 6.2.1. Let $\tilde{X}$ be a finite-dimensional CAT(0) cube complex and $\mathcal{H}$ denote the combinatorial pocset of $\tilde{X}$. Let Y be a convex subcomplex of $\tilde{X}$. Let $\mathcal{I}(Y)$ be the set of halfspaces, which contain Y, and $\mathcal{O}(Y) := \{\mathfrak{h}^* \,|\, \mathfrak{h} \in \mathcal{I}(Y)\}$. Then $Y = \bigcap_{\mathfrak{h} \in \mathcal{I}(Y)} \mathfrak{h}$ by Proposition 5.4.15.

If $\mathfrak{h}$ is a halfspace which does not belong to $\mathcal{I}(Y) \cup \mathcal{O}(Y)$, then both $\mathfrak{h}$ and $\mathfrak{h}^*$ intersect Y. So, by Lemma 5.2.20, the hyperplane of $\mathfrak{h}$ passes through Y. Conversely, if a hyperplane intersects Y, then its halfspaces $\mathfrak{h}$ and $\mathfrak{h}^*$ intersect Y simultaneously. So $\mathfrak{h}$ and $\mathfrak{h}^*$ are contained in neither $\mathcal{I}(Y)$ nor $\mathcal{O}(Y)$.

Let $\mathcal{H}(Y)$ denote $\mathcal{H} \setminus (\mathcal{I}(Y) \cup \mathcal{O}(Y))$. By the above, $\mathcal{H}(Y)$ is equal to $\{\mathfrak{h} \in \mathcal{H} \,|\, \text{the hyperplane of } \mathfrak{h} \text{ passes through } Y\}$. Then $\mathcal{H}(Y)$ is a subpocset of $\mathcal{H}$. Precisely, for each $\mathfrak{h} \in \mathcal{H}(Y)$, we have $\mathfrak{h}^* \in \mathcal{H}(Y)$.

Definition 6.2.2. Let $\tilde{X}$ be a CAT(0) cube complex and Y a convex subcomplex of $\tilde{X}$. If $\mathcal{H}$ is the combinatorial pocset of $\tilde{X}$, then we call the subpocset $\mathcal{H}(Y) := \{\mathfrak{h} \in \mathcal{H} \,|\, \text{the hyperplane } \hat{\mathfrak{h}} \text{ of } \mathfrak{h} \text{ passes through } Y\}$ the *combinatorial subpocset generated by* Y. The set $\mathcal{I}(Y) := \{\mathfrak{h} \in \mathcal{H} \,|\, Y \subset \mathfrak{h}\}$ is called the *partial inward orientation* of Y. And $\mathcal{O}(Y) := \{\mathfrak{h} \,|\, \mathfrak{h} \in \mathcal{I}(Y)\}$ is said to be the *partial outward orientation* of Y.

Lemma 6.2.3 (Lemma 2.6 in [12])**.** *Let Y be a convex subcomplex of a finite-dimensional* CAT(0) *cube complex $\tilde{X}$. Let $\pi_{\mathcal{H}(Y)}$ be the collapsing from $\tilde{X}$ to the dual cube complex $\tilde{X}(\mathcal{H}(Y))$ of the combinatorial subpocset $\mathcal{H}(Y)$.*

1. *There is an injective cubical map $\iota_Y : \tilde{X}(\mathcal{H}(Y)) \hookrightarrow \tilde{X}$ such that the image of $\tilde{X}(\mathcal{H}(Y))$ is Y and $\mathcal{I}(\iota_Y(y)) = \mathcal{I}(Y) \cup \mathcal{I}(y)$ for all vertices $y \in \tilde{X}(\mathcal{H}(Y))$. In particular, $\tilde{X}(\mathcal{H}(Y))$ is isomorphic to Y.*

2. *Let $\mathcal{H}'$ be a subpocset of $\mathcal{H}$ satisfying that, whenever $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2 \supsetneq \mathfrak{h}_3$ and $\{\mathfrak{h}_1, \mathfrak{h}_3\} \subset \mathcal{H}'$, we have $\mathfrak{h}_2 \in \mathcal{H}'$. Then there is a convex subcomplex Y such that the convex subpocset of Y is exactly $\mathcal{H}'$.*

Lemma 6.2.4. *Let Y be a convex subcomplex of a $\mathrm{CAT}(0)$ cube complex $\tilde{X}$.*

1. *The composition $\pi_Y = \iota_Y \circ \pi_{\mathcal{H}(Y)} : \tilde{X} \to \tilde{X}$ satisfies that, for every vertex x of $\tilde{X}$, the inward orientation of $\pi_Y(x)$ is $\mathcal{I}(Y) \cup (\mathcal{I}(x) \cap \mathcal{H}(Y))$.*

2. *The cubical map π_Y defined in (1) is the closest-point projection in Lemma 5.4.21.*

3. *If $Z \subseteq Y$ are convex subcomplexes of $\tilde{X}$, and if π_Y and π_Z are the projections from $\tilde{X}$ to Y and Z, respectively, then $\pi_Z \pi_Y(x) = \pi_Z(x)$ for every vertex x in $\tilde{X}$.*

Proof. (1) For each vertex x of $\tilde{X}$, the inward orientation of $\pi_{\mathcal{H}(Y)}(x)$ is $\mathcal{I}(x) \cap \mathcal{H}(Y)$ by Proposition 6.1.2. So, by Lemma 6.2.3.(1), the inward orientation of $\pi_Y(x)$ is $\mathcal{I}(Y) \cup \mathcal{I}(\pi_{\mathcal{H}(Y)}(x)) = \mathcal{I}(Y) \cup (\mathcal{I}(x) \cap \mathcal{H}(Y))$.

(2) Let x be a vertex of $\tilde{X}$. Then, by Remark 5.4.9,

$$\begin{aligned} d(x, \pi_Y(x)) &= |\mathcal{I}(\pi_Y(x)) - \mathcal{I}(x)| \\ &= |\mathcal{I}(Y) - \mathcal{I}(x)| \\ &= |\{\mathfrak{h} \in \mathcal{H} \mid x \in \mathfrak{h}^*, Y \subseteq \mathfrak{h}\}| = d(x, Y). \end{aligned}$$

So π_Y is as same as the closest-point projection in Lemma 5.4.21.

(3) Since Z is a convex subcomplex of Y, the subpocset $\mathcal{H}(Z)$ generated by Z is contained in the subpocset $\mathcal{H}(Y)$. And $\mathcal{I}(Z) \supseteq \mathcal{I}(Y)$ for two partial inward orientations of Y and Z. Then, for each vertex x of $\tilde{X}$, the inward orientation of $\pi_Y(x)$ is $\mathcal{I}(Y) \cup (\mathcal{I}(x) \cap \mathcal{H}(Y))$ by Lemma 6.2.4.(1). If $\mathcal{I}(\pi_Z(x))$ and $\mathcal{I}(\pi_Z\pi_Y(x))$ are the inward orientations of $\pi_Z(x)$ and $\pi_Z\pi_Y(x)$, respectively, then

$$\mathcal{I}(\pi_Z(x)) = \mathcal{I}(Z) \cup (\mathcal{I}(x) \cap \mathcal{H}(Y))$$

and

$$\begin{aligned} \mathcal{I}(\pi_Z\pi_Y(x)) &= \mathcal{I}(Z) \cup (\mathcal{I}(\pi_Y(x)) \cap \mathcal{H}(Z)) \\ &= \mathcal{I}(Z) \cup \big[\big(\mathcal{I}(Y) \cup (\mathcal{I}(x) \cap \mathcal{H}(Y))\big) \cap \mathcal{H}(Z)\big] \end{aligned}$$

$$= \mathcal{I}(Z) \cup (\mathcal{I}(x) \cap \mathcal{H}(Z)) = \mathcal{I}(\pi_Z(x)).$$

Then, by Remark 5.3.5.(2), the vertex $\pi_Z \pi_Y(x)$ is equal to $\pi_Z(x)$. $\square$

Lemma 6.2.5. *Let $\tilde{X}$ be a finite-dimensional* CAT(0) *cube complex and $\mathcal{H}$ the combinatorial pocset of $\tilde{X}$. If Y_1 and Y_2 are convex subcomplexes of $\tilde{X}$ such that $Y_1 \cap Y_2$ is nonempty, then the following hold.*

1. *A hyperplane $\hat{\mathfrak{h}}$ passes through both Y_1 and Y_2 if and only if $\hat{\mathfrak{h}}$ passes through $Y_1 \cap Y_2$.*
2. *The combinatorial subpocset $\mathcal{H}(Y_1 \cap Y_2)$ generated by $Y_1 \cap Y_2$ is the intersection of $\mathcal{H}(Y_1)$ and $\mathcal{H}(Y_2)$.*
3. *The partial inward orientation $\mathcal{I}(Y_1 \cap Y_2)$ is equal to $\mathcal{I}(Y_1) \cup \mathcal{I}(Y_2)$.*
4. *If y is a vertex of Y_1 and $\mathcal{I}(y)$ is the inward orientation of y, then $\mathcal{I}(Y_1) \subseteq \mathcal{I}(y)$ and $\mathcal{I}(y) \subseteq \mathcal{I}(Y_1) \cup \mathcal{H}(Y_1)$.*
5. *If π_{Y_2} and $\pi_{Y_1 \cap Y_2}$ are the projections to Y_2 and $Y_1 \cap Y_2$, respectively, then $\pi_{Y_2}(y) = \pi_{Y_1 \cap Y_2}(y)$ for all vertex $y \in Y_1$.*

Proof. (1) If $\hat{\mathfrak{h}}$ passes through $Y_1 \cap Y_2$, then it is obvious that $\hat{\mathfrak{h}}$ passes through both Y_1 and Y_2. Assume that a hyperplane $\hat{\mathfrak{h}}$ passes through both Y_1 and Y_2. If $\mathfrak{h}$ is a halfspace of $\hat{\mathfrak{h}}$, then the following four intersections are nonempty:

$$\mathfrak{h} \cap Y_1, \quad \mathfrak{h} \cap Y_2, \quad \mathfrak{h}^* \cap Y_1, \quad \mathfrak{h}^* \cap Y_2.$$

By Lemma 5.4.17, the intersections $\mathfrak{h} \cap Y_1 \cap Y_2$ and $\mathfrak{h}^* \cap Y_1 \cap Y_2$ are nonempty. Hence, by Lemma 5.2.20, the hyperplane $\hat{\mathfrak{h}}$ passes through $Y_1 \cap Y_2$.

(2) It is an immediate result from (1).

(3) By Proposition 5.4.15.(2),

$$\begin{aligned} Y_1 \cap Y_2 &= \left(\bigcap \{\mathfrak{h} \mid \mathfrak{h} \in \mathcal{I}(Y_1)\}\right) \cap \left(\bigcap \{\mathfrak{h} \mid \mathfrak{h} \in \mathcal{I}(Y_2)\}\right) \\ &= \bigcap_{\mathfrak{h} \in \mathcal{I}(Y_1) \cup \mathcal{I}(Y_2)} \mathfrak{h}. \end{aligned}$$

So $\mathcal{I}(Y_1) \cup \mathcal{I}(Y_2) \subseteq \mathcal{I}(Y_1 \cap Y_2)$. For every halfspace $\mathfrak{h}$ in $\mathcal{I}(Y_1 \cap Y_2)$, the hyperplane $\hat{\mathfrak{h}}$ of $\mathfrak{h}$ does not intersect $Y_1 \cap Y_2$. So, by (1), we have $\hat{\mathfrak{h}}$ passes through neither Y_1 nor Y_2. Then either Y_1 or Y_2 is contained in $\mathfrak{h}$. So $\mathfrak{h} \in \mathcal{I}(Y_1) \cup \mathcal{I}(Y_2)$. Therefore, $\mathcal{I}(Y_1 \cap Y_2) = \mathcal{I}(Y_1) \cup \mathcal{I}(Y_2)$.

(4) Every halfspace in $\mathcal{I}(Y_1)$ contains y because $y \in Y_1$. So we have $\mathcal{I}(Y_1) \subset \mathcal{I}(y)$. Let $\mathfrak{h} \in \mathcal{I}(y)$ be given. Since $\mathfrak{h}$ intersects Y_1, either $\mathfrak{h} \in \mathcal{I}(Y_1)$ or $\mathfrak{h} \in \mathcal{H}(Y_1)$. Hence, $\mathcal{I}(y) \subset \mathcal{I}(Y_1) \cup \mathcal{H}(Y_1)$.

(5) By Lemma 6.2.4.(1), the inward orientation of $\pi_{Y_1 \cap Y_2}(y)$, denoted by $\mathcal{I}(\pi_{Y_1 \cap Y_2}(y))$, is $\mathcal{I}(Y_1 \cap Y_2) \cup (I(y) \cap \mathcal{H}(Y_1 \cap Y_2))$. Then, by (3), (2) and (4),

$$\begin{aligned}
\mathcal{I}(\pi_{Y_1 \cap Y_2}(y)) &= \mathcal{I}(Y_1) \cup \mathcal{I}(Y_2) \cup \left[\mathcal{I}(y) \cap \mathcal{H}(Y_1) \cap \mathcal{H}(Y_2)\right] \\
&= \mathcal{I}(Y_2) \cup \left[\mathcal{I}(y) \cap (\mathcal{I}(Y_1) \cup \mathcal{H}(Y_2))\right] \\
&= \mathcal{I}(Y_1) \cup \left[\mathcal{I}(Y_2) \cup (\mathcal{I}(y) \cap \mathcal{H}(Y_2))\right] \\
&= \mathcal{I}(Y_1) \cup \mathcal{I}(\pi_{Y_2}(y))
\end{aligned}$$

where $\mathcal{I}(\pi_{Y_2}(y))$ is the inward orientation of $\pi_{Y_2}(y)$. Then $\mathcal{I}(\pi_{Y_1 \cap Y_2}(y)) \supseteq \mathcal{I}(\pi_{Y_2}(y))$. Since both $\mathcal{I}(\pi_{Y_1 \cap Y_2}(y))$ and $\mathcal{I}(\pi_{Y_2}(y))$ are principal ultrafilters of $\mathcal{H}$, we have $\mathcal{I}(\pi_{Y_1 \cap Y_2}(y)) = \mathcal{I}(\pi_{Y_2}(y))$. (Precisely, if a halfspace $\mathfrak{h}$ exists in $\mathcal{I}(\pi_{Y_1 \cap Y_2}(y)) \setminus \mathcal{I}(\pi_{Y_2}(y))$, then $\mathfrak{h}^* \in \mathcal{I}(\pi_{Y_2}(y))$ so that $\{\mathfrak{h}, \mathfrak{h}^*\} \subset \mathcal{I}(\pi_{Y_1 \cap Y_2}(y))$. This is a contradiction.) Therefore, $\pi_{Y_1 \cap Y_2}(y) = \pi_{Y_2}(y)$ by Remark 5.5.5. $\square$

6.3 Bridges

Definition 6.3.1. For disjoint halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of a CAT(0) cube complex $\tilde{X}$, the *bridge* $B(\mathfrak{h}_1, \mathfrak{h}_2)$ between $\mathfrak{h}_1$ and $\mathfrak{h}_2$ is the smallest convex subcomplex of $\tilde{X}$ containing all pairs of vertices $x \in \mathfrak{h}_1$ and $y \in \mathfrak{h}_2$ satisfying $d(x, y) = d(\mathfrak{h}_1, \mathfrak{h}_2)$ where d is the combinatorial metric on $\tilde{X}$.

Remark 6.3.2. Behrstock-Charney in [1] introduce this concept for the universal covers of RAAGs. Chatterji-Fernós-Iozzi in [12, Section 2.G] develop it for arbitrary finite dimensional CAT(0) cube complexes.[2]

Lemma 6.3.3 (Lemma 2.18 in [12])**.** *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be disjoint halfspaces of a finite-dimensional* CAT(0) *cube complex $\tilde{X}$. Let $\mathcal{H}$ be the combinatorial pocset of $\tilde{X}$. If $B := B(\mathfrak{h}_1, \mathfrak{h}_2)$ is the bridge between $\mathfrak{h}_1$ and $\mathfrak{h}_2$, then the following hold: see Figure 6.1.*

1. *Every hyperplane $\hat{\mathfrak{h}}$ passing through B satisfies one of the following statements.*

[2]Our notation is different from one of [12]. In this paper, the notation $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ has another meaning.

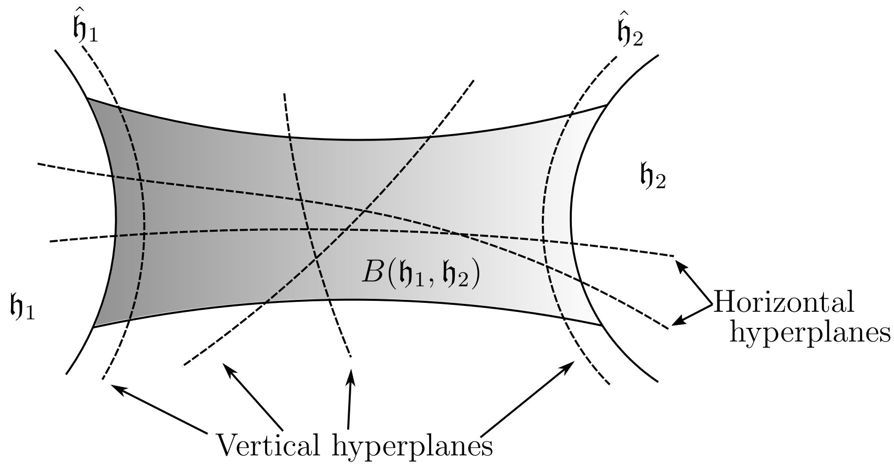

Figure 6.1: The bridge $B(\mathfrak{h}_1, \mathfrak{h}_2)$ between disjoint halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$

(a) $\hat{\mathfrak{h}}$ crosses both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$.

(b) $\hat{\mathfrak{h}}$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$.

2. *If $\mathcal{H}(B)$ is the subpocset of $\mathcal{H}$ generated by B, then both $\mathcal{H}_h(B) := \{\mathfrak{h} \in \mathcal{H}(B) \mid \hat{\mathfrak{h}}$ crosses $\hat{\mathfrak{h}}_1\}$ and $\mathcal{H}_v(B) := \{\mathfrak{h} \in \mathcal{H}(B) \mid \hat{\mathfrak{h}}$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_2\}$ are subpocsets of $\mathcal{H}$.*

3. *If $\tilde{X}_h$ and $\tilde{X}_v$ are the dual cube complexes of $\mathcal{H}_h(B)$ and $\mathcal{H}_v(B)$, respctively, then B is isomorphic to $\tilde{X}_h \times \tilde{X}_v$.*

Definition 6.3.4. We call the hyperplanes of halfspaces in $\mathcal{H}_v(B)$ of the above lemma *vertical hyperplanes*. And the hyperplanes of halfspaces in $\mathcal{H}_h(B)$ of the above lemma are called *horizontal hyperplanes*.

Remark 6.3.5. In the setting of Lemma 6.3.3, the hyperplanes $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are vertical hyperplanes of $B(\mathfrak{h}_1, \mathfrak{h}_2)$. If $x_1 \in \mathfrak{h}_1$ and $x_2 \in \mathfrak{h}_2$ are vertices such that $d(x_1, x_2) = d(\mathfrak{h}_1, \mathfrak{h}_2)$, then $\tilde{X}_v$ is isomorphic to the smallest convex subcomplex containing $\{x_1, x_2\}$. And $\tilde{X}_h$ is isomorphic to $B(\mathfrak{h}_1, \mathfrak{h}_2) \cap \mathfrak{h}_1$.

By Lemma 5.4.23, a hyperplane $\hat{\mathfrak{h}}$ is a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ if and only if $\hat{\mathfrak{h}}$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$. Note that Chatterji-Fernós-Iozzi [12] did not answer the question whether every hyperplane crossing both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ passes through the bridge $B(\mathfrak{h}_1, \mathfrak{h}_2)$. We will prove that this statement is true in Proposition 6.3.13.

Lemma 6.3.6 (Lemma 2.18 in [12]). *In the setting of Lemma 6.3.3, a hyperplane $\hat{\mathfrak{h}}$ is a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ if and only if there are vertices $x_1, y_1 \in \mathfrak{h}_1$ and $x_2, y_2 \in \mathfrak{h}_2$ such that $d(x_1, x_2) = d(y_1, y_2) = d(\mathfrak{h}_1, \mathfrak{h}_2)$ and $\hat{\mathfrak{h}}$ separates the convex hull of $\{x_1, x_2\}$ from the convex hull of $\{y_1, y_2\}$.*

We will prove the equivalent condition for horizontal hyperplanes, which is stronger than Lemma 6.3.6: see Proposition 6.3.13.

Example 4. For every halfspace $\mathfrak{h}$, the bridge $B(\mathfrak{h}, \mathfrak{h}^*)$ is equal to the carrier of $\hat{\mathfrak{h}}$ for every halfspace $\mathfrak{h}$.

6.3.1 Bridges between disjoint hyperplanes

Now we define the bridge between two hyperplanes.

Definition 6.3.7. If $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are disjoint hyperplanes, then the *bridge* between $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$, denoted by $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is the smallest convex subcomplex which contains all pairs of vertices $x \in \mathcal{N}(\hat{\mathfrak{h}}_1)$ and $y \in \mathcal{N}(\hat{\mathfrak{h}}_2)$ satisfying $d(x, y) = d(\mathcal{N}(\hat{\mathfrak{h}}_1), \mathcal{N}(\hat{\mathfrak{h}}_2))$.

Lemma 6.3.8. *If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are disjoint halfspaces of a finite-dimensional* CAT(0) *cube complex $\tilde{X}$ and $\mathfrak{h}_1 \neq \mathfrak{h}_2^*$, then $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2) = B(\mathfrak{h}_1, \mathfrak{h}_2) \cap \mathfrak{h}_1^* \cap \mathfrak{h}_2^*$.*

Proof. Choose two vertices $x_1 \in \mathcal{N}(\hat{\mathfrak{h}}_1)$ and $x_2 \in \mathcal{N}(\hat{\mathfrak{h}}_2)$ satisfying that $d(x_1, x_2) = d(\mathcal{N}(\hat{\mathfrak{h}}_1), \mathcal{N}(\hat{\mathfrak{h}}_2))$. By the hypothesis, $\mathcal{N}(\hat{\mathfrak{h}}_1) \subset \mathfrak{h}_2^*$ and $\mathcal{N}(\hat{\mathfrak{h}}_2) \subset \mathfrak{h}_1^*$. So $x_1 \in \mathfrak{h}_2^*$ and $x_2 \in \mathfrak{h}_1^*$.

Because x_1 is the vertex of $\hat{\mathfrak{h}}_1$ closest to x_2, we have $x_1 = \pi_{\mathcal{N}(\hat{\mathfrak{h}}_1)}(x_2)$ where $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_1)}$ is the closest-point projection to $\mathcal{N}(\hat{\mathfrak{h}}_1)$. Then, by Lemma 6.2.5.(5), we have $x_1 \in \mathcal{N}(\hat{\mathfrak{h}}_1) \cap \mathfrak{h}_2^*$. Similarly, $x_2 \in \mathcal{N}(\hat{\mathfrak{h}}_2) \cap \mathfrak{h}_1^*$. So $\{x_1, x_2\} \subset \mathfrak{h}_1^* \cap \mathfrak{h}_2^*$. Since x_1 and x_2 are arbitrary, the bridge $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is contained in $\mathfrak{h}_1^* \cap \mathfrak{h}_2^*$.

For each $j = 1, 2$, let e_j be the edge which contains x_j and intersects $\hat{\mathfrak{h}}_j$. And let y_j be the vertex of e_j which belongs to $\mathfrak{h}_j$. Let $\vec{e}_1$ and $\vec{e}_2$ be the oriented edges of e_1 and e_2 whose terminal halfspaces are $\mathfrak{h}_1^*$ and $\mathfrak{h}_2$, respectively. If $\vec{L} = (\vec{\mathsf{e}}_1, \ldots, \vec{\mathsf{e}}_m)$ is a geodesic path from x_1 to x_2, then $\vec{L}' = (\vec{e}_1, \vec{\mathsf{e}}_1, \ldots, \vec{\mathsf{e}}_m, \vec{e}_2)$ is a combinatorial path from y_1 to y_2. If $\hat{\mathfrak{h}}$ is a hyperplane seperating y_1 and y_2, then it passes through an edge of $\vec{L}'$. So $\hat{\mathfrak{h}}$ is either $\hat{\mathfrak{h}}_1$, $\hat{\mathfrak{h}}_2$ or a hyperplane seperating x_1 from x_2. Note that a hyperplane seperating x_1 from x_2 also separates $\mathcal{N}(\hat{\mathfrak{h}}_1)$ from $\mathcal{N}(\hat{\mathfrak{h}}_2)$ by Lemma 5.4.23. It implies that every hyperplane seperating y_1 from y_2 also separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$. So

$d(y_1, y_2) = d(\mathfrak{h}_1, \mathfrak{h}_2)$ and the combinatorial path $\vec{L}'$ is a geodesic path from y_1 to y_2. By the definition of bridge, $y_1, y_2 \in B(\mathfrak{h}_1, \mathfrak{h}_2)$. Because $B(\mathfrak{h}_1, \mathfrak{h}_2)$ is convex, every edge of $\vec{L}$ lies in $B(\mathfrak{h}_1, \mathfrak{h}_2)$. Since we choose x_1 and x_2 arbitrarily, the bridge $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is contained in $B(\mathfrak{h}_1, \mathfrak{h}_2)$. So $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2) \subseteq B(\mathfrak{h}_1, \mathfrak{h}_2) \cap \mathfrak{h}_1^* \cap \mathfrak{h}_2^*$. It implies that, if a hyperplane passes through $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$, then it passes through $B(\mathfrak{h}_1, \mathfrak{h}_2)$, that is, it is either a vertical hyperplane or a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$.

Let $\hat{\mathfrak{h}}_v$ be a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ which is neither $\hat{\mathfrak{h}}_1$ nor $\hat{\mathfrak{h}}_2$. Since $\hat{\mathfrak{h}}_v$ does not intersect $\mathfrak{h}_j$ for each $j = 1, 2$, we have $\hat{\mathfrak{h}}_v \subset \mathfrak{h}_1^* \cap \mathfrak{h}_2^*$. Then $\hat{\mathfrak{h}}_v$ cross neither $\hat{\mathfrak{h}}_1$ nor $\hat{\mathfrak{h}}_2$ so that a halfspace $\mathfrak{h}_v$ of $\hat{\mathfrak{h}}_v$ contains $\mathcal{N}(\hat{\mathfrak{h}}_1)$ and its opposite halfspace contains $\mathcal{N}(\hat{\mathfrak{h}}_2)$. That is, $\hat{\mathfrak{h}}_v$ separates $\mathcal{N}(\hat{\mathfrak{h}}_1)$ from $\mathcal{N}(\hat{\mathfrak{h}}_2)$. So $\hat{\mathfrak{h}}_v$ passes through $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$.

Let $\hat{\mathfrak{h}}_h$ be a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$. Choose a halfspace $\mathfrak{h}_h$ of $\hat{\mathfrak{h}}_h$. By Lemma 6.3.6, there exist a pair of vertices $y_1 \in \mathfrak{h}_h \cap \mathfrak{h}_1$ and $y_2 \in \mathfrak{h}_h \cap \mathfrak{h}_2$ such that $d(y_1, y_2) = d(\mathfrak{h}_1, \mathfrak{h}_2)$. Let $\vec{L} = (\vec{e}_1, \ldots, \vec{e}_n)$ be a geodesic path from y_1 to y_2. Then every edge of $\vec{L}$ belongs to $\mathfrak{h}_h$ because of convexity. Note that the hyperplane $\hat{\mathfrak{h}}$ passing through $\vec{e}_1$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$ by Lemma 5.4.23. Since $y_1 \in \vec{e}_1$ and $\vec{e}_1$ is not contained in $\mathfrak{h}_1$, the carrier of $\hat{\mathfrak{h}}$ intersects both $\mathfrak{h}_1$ and $\mathfrak{h}_1^*$. Because $\hat{\mathfrak{h}}$ does not cross $\hat{\mathfrak{h}}_1$, the only possibility is that $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_1$, i.e., $\hat{\mathfrak{h}}_1$ passes through $\vec{e}_1$. Similarly, the hyperplane $\hat{\mathfrak{h}}_2$ passes through $\vec{e}_n$. So we have $x_1 \in \mathcal{N}(\hat{\mathfrak{h}}_1)$ and $x_2 \in \mathcal{N}(\hat{\mathfrak{h}}_2)$. Note that $(\vec{e}_2, \ldots, \vec{e}_{n-1})$ is the geodesic path from x_1 to x_2. Since the hyperplane passing through each $\vec{e}_j$ is a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$, it cross neither $\hat{\mathfrak{h}}_1$ nor $\hat{\mathfrak{h}}_2$. So $d(x_1, x_2) = d(\mathcal{N}(\hat{\mathfrak{h}}_1), \mathcal{N}(\hat{\mathfrak{h}}_2))$ by Lemma 5.4.23. Then we have $\{x_1, x_2\} \in B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ by definition, and $\mathfrak{h}_h$ intersects $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$. Because $\mathfrak{h}_h$ is arbitrary halfspace of $\hat{\mathfrak{h}}_h$, the opposite halfspace of $\mathfrak{h}_h$ also intersects $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$. Hence, $\hat{\mathfrak{h}}_h$ passes through the bridge $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ by Lemma 5.2.20.

In conclusion, a hyperplane $\hat{\mathfrak{h}}$ passes through $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ if and only if either

- $\hat{\mathfrak{h}}$ is a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$, which is not any of $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$, or
- $\hat{\mathfrak{h}}$ is a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$.

So every halfspace containing $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is either $\mathfrak{h}_1^*$, $\mathfrak{h}_2^*$ or a halfspace containing $B(\mathfrak{h}_1, \mathfrak{h}_2)$. Therefore, by Proposition 5.4.15,

$$B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2) = \mathfrak{h}_1^* \cap \mathfrak{h}_2^* \cap \left(\bigcap \{\mathfrak{h} \in \mathcal{H} \mid B(\mathfrak{h}_1, \mathfrak{h}_2) \subset \mathfrak{h}\} \right)$$

$$= \mathfrak{h}_1^* \cap \mathfrak{h}_2^* \cap B(\mathfrak{h}_1, \mathfrak{h}_2)$$

where $\mathcal{H}$ is the set of all halfspaces of the CAT(0) cube complex $\tilde{X}$. $\square$

Remark 6.3.9. Due to Lemma 6.3.8, bridges between two hyperplanes share many properties with the bridge of disjoint halfspaces. For example, every hyperplane passing through the bridge $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ either separates $\mathcal{N}(\hat{\mathfrak{h}}_1)$ from $\mathcal{N}(\hat{\mathfrak{h}}_2)$, or crosses both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$. So we can define vertical and horizontal hyperplanes for $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$. In fact, every horizontal hyperplane of $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is also a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$, and every vertical hyperplane of $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ which is neither $\hat{\mathfrak{h}}_1$ nor $\hat{\mathfrak{h}}_2$.

Moreover, $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ can be decomposed into the product of two convex subcomplexes. One is the intersection of $\mathcal{N}(\hat{\mathfrak{h}}_1)$ and $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$, and the other is the convex hull of two vertices $x_1 \in \mathcal{N}(\hat{\mathfrak{h}}_1)$ and $x_2 \in \mathcal{N}(\hat{\mathfrak{h}}_2)$ such that $d(x_1, x_2) = d(\mathcal{N}(\hat{\mathfrak{h}}_1), \mathcal{N}(\hat{\mathfrak{h}}_2))$.

Remark 6.3.10. The reason why it is the bridge is explained by [12, Lemma 2.24]: For disjoint halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$, if $x_1 \in \mathfrak{h}_1$ and $x_2 \in \mathfrak{h}_2$, then

$$d(x_1, x_2) = d(x_1, \pi_B(x_1)) + d(\pi_B(x_1), \pi_B(x_2)) + d(\pi_B(x_2), x_2)$$

where $\pi_B = \pi_{B(\mathfrak{h}_1, \mathfrak{h}_2)}$ is the closest-point projection to the bridge $B(\mathfrak{h}_1, \mathfrak{h}_2)$. We modify this statement to fit into the bridges of hyperplanes.

Lemma 6.3.11. *Let $\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n$ be a finite sequence of halfspaces of a finite-dimensional* CAT(0) *cube complex. And let a pair of vertices $x \in \mathfrak{h}_1^*$ and $y \in \mathfrak{h}_n$ be given. Let B_j denote the bridge between $\hat{\mathfrak{h}}_j$ and $\hat{\mathfrak{h}}_{j+1}$ for each $j \in \{1, \dots n-1\}$. Then the following hold.*[3]

1. *For each $j = 0, \dots, n-1$, let y_j be the image of x under the closest-point projection $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1})}$. Then y_j is equal to $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cup \mathfrak{h}_{j+1}}(x)$.*

2. *For each $j = 1, \dots, n-1$, the vertex y_j is contained in $B_j \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})$.*

3. *Let x_j be the image of x under the closest-point projection to B_j for each $j = 1, \dots, n-1$. Then x_j is contained in $B_j \cap \mathcal{N}(\hat{\mathfrak{h}}_j)$.*

4. *Let x_n denote $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_n)}(y)$. If $x_0 := x$ and $y_n := y$, then, for each $j = 0, \dots, n-1$, we have $d(y_j, y_{j+1}) = d(y_j, x_{j+1}) + d(x_{j+1}, y_{j+1})$.*

[3]See Figure 6.2.

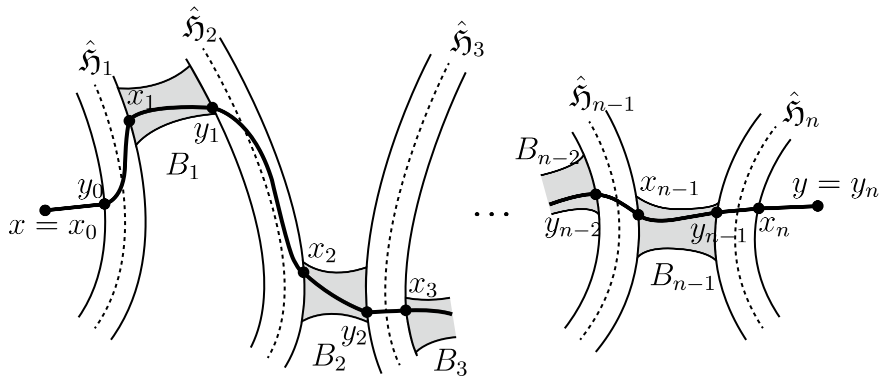

Figure 6.2: If $\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n$ is a sequence of halfspaces, $x \in \mathfrak{h}_1^*$ and $y \in \mathfrak{h}_n$, then there is a combinatorial geodesic joining x and y which crosses the bridges of $B(\hat{\mathfrak{h}}_j, \hat{\mathfrak{h}}_{j+1})$.

5. *The following equality holds.*

$$\begin{aligned} d(x, y) &= d(x_0, y_0) + d(y_0, x_1) + d(x_1, y_1) + \cdots + d(y_{n-1}, x_n) + d(x_n, y_n) \\ &= \sum_{i=0}^{n} d(x_i, y_i) + \sum_{i=0}^{n-1} d(y_i, x_{i+1}). \end{aligned}$$

Proof. (1) Fix $j \in \{0, \ldots, n-1\}$. Since $x \in \mathfrak{h}_{j+1}^*$, the vertex y_j is contained in $\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cap \mathfrak{h}_{j+1}^*$ by Lemma 6.2.5.(5). Then y_j is equal to $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cap \mathfrak{h}_{j+1}^*}(x)$ by Lemma 6.2.4.(3). Note that $\mathfrak{h}_{j+1}^* \cap (\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cup \mathfrak{h}_{j+1}) = \mathfrak{h}_{h+1}^* \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})$. So, by Lemma 6.2.5.(5) and Lemma 6.2.4.(3), we have $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cup \mathfrak{h}_{j+1}}(x) \in \mathfrak{h}_{j+1}^* \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})$ and $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cup \mathfrak{h}_{j+1}}(x) = \pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1}) \cap \mathfrak{h}_{j+1}^*}(x) = y$.

(2) For each $j = 1, \ldots, n-1$, let z_j be the vertex $\pi_{\mathcal{N}(\hat{\mathfrak{h}}_j)}(y_j)$. To prove that y_j is contained in B_j, we need only to show that $d(y_j, z_j)$ is equal to the distance between $\mathcal{N}(\hat{\mathfrak{h}}_j)$ and $\mathcal{N}(\hat{\mathfrak{h}}_{j+1})$. We will actually show that every hyperplane $\hat{\mathfrak{h}}$ seperating y_j from z_j also separates $\mathcal{N}(\hat{\mathfrak{h}}_j)$ from $\mathcal{N}(\hat{\mathfrak{h}}_{j+1})$. First, note that $\hat{\mathfrak{h}}$ does not pass through $\mathcal{N}(\hat{\mathfrak{h}}_j)$ by Lemma 5.4.21.(3). By Lemma 5.4.21.(2), the combinatorial distance between x and y_j is equal to $d(x, z_j) + d(z_j, y_j)$. So $\hat{\mathfrak{h}}$ separates x from y_j. Because $y_j = \pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(x)$, the hyperplane $\hat{\mathfrak{h}}$ separates x from $\mathcal{N}(\hat{\mathfrak{h}}_{j+1})$ by Lemma 5.4.21.(3). Then $\hat{\mathfrak{h}}$ does not pass

through $\mathcal{N}(\hat{\mathfrak{h}}_{j+1})$. Therefore, $\hat{\mathfrak{h}}$ separates $\mathcal{N}(\hat{\mathfrak{h}}_j)$ from $\mathcal{N}(\hat{\mathfrak{h}}_{j+1})$.

(3) Note that $x \in \mathcal{N}(\hat{\mathfrak{h}}_j) \cup \mathfrak{h}_j^*$. Since $\mathcal{N}(\hat{\mathfrak{h}}_j) \cup \mathfrak{h}_j^*$ is convex by Lemma 5.4.19, we have $x_j \in B_j \cap (\mathcal{N}(\hat{\mathfrak{h}}_j) \cup \mathfrak{h}_j^*)$ by Lemma 6.2.5.(5). Because B_j is a subcomplex of $\mathfrak{h}_j$, we have $B_j \cap (\mathcal{N}(\hat{\mathfrak{h}}_j) \cup \mathfrak{h}_j^*) = B_j \cap \mathcal{N}(\hat{\mathfrak{h}}_j)$ so that $x_j \in B_j \cap \mathcal{N}(\hat{\mathfrak{h}}_j)$.

(4) If $j = n-1$, it is immediately obtained from Lemma 5.4.21.(2) that $d(y_{n-1}, y_n) = d(y_{n-1}, x_n) + d(x_n, y_n)$. Fix $j \in \{0, \ldots, n-2\}$. By (3) and Lemma 6.2.4.(3), we have

$$x_{j+1} = \pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(x_{j+1}) = \pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})} \pi_{B_{j+1}}(x) = \pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(x).$$

On the other hand, $\pi_{B_{j+1}}(y_j)$ is equal to $\pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(y_j)$ by Lemma 6.2.5.(5). Then, applying Lemma 6.2.4.(3),

$$\pi_{B_{j+1}}(y_j) = \pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})} \pi_{\mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(x) = \pi_{B_{j+1} \cap \mathcal{N}(\hat{\mathfrak{h}}_{j+1})}(x) = x_{j+1}.$$

Note that $y_{j+1} \in B_{j+1}$ by (2). Hence, by Lemma 5.4.21.(2), we have $d(y_j, y_{j+1}) = d(y_j, x_{j+1}) + d(x_{j+1}, y_{j+1})$.

(5) Because $\mathcal{N}(\hat{\mathfrak{h}}_j) \subset \mathfrak{h}_i$ whenever $i < j$, we have $\mathcal{N}(\hat{\mathfrak{h}}_1) \cup \mathfrak{h}_1 \supsetneq \mathcal{N}(\hat{\mathfrak{h}}_2) \cup \mathfrak{h}_2 \supsetneq \cdots \supsetneq \mathcal{N}(\hat{\mathfrak{h}}_n) \cup \mathfrak{h}_n$. Then, by (1) and Lemma 5.4.21.(2), $d(x, y) = d(x_0, y_0) + d(y_0, y_1) + \cdots + d(y_{n-1}, y_n)$. Hence, applying (4) to each $d(y_j, y_{j+1})$, we have $d(x, y) = d(x_0, y_0) + d(y_0, x_1) + d(x_1, y_1) + \cdots + d(y_{n-1}, x_n) + d(x_n, y_n)$. □

6.3.2 Criterion for horizontal hyperplanes

[4]

Remark 6.3.12. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be disjoint halfspaces. If a hyperplane $\hat{\mathfrak{h}}$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$, then it must passes through the bridge $B(\mathfrak{h}_1, \mathfrak{h}_2)$ so that it is a vertical hyperplane. That is, $\hat{\mathfrak{h}}$ is a vertical hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ if and only if it separates $\mathfrak{h}_1$ from $\mathfrak{h}_2$.

Similarly to vertical hyperplanes, we are going to give a criterion for horizontal hyperplanes.

[4] This section is not used for our main result. So you can skip it if you do not focus on bridges.

Proposition 6.3.13. *For every disjoint halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$, a hyperplane $\hat{\mathfrak{h}}$ is a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$ if and only if $\hat{\mathfrak{h}}$ crosses both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$. Moreover, if $\mathfrak{h}_1$ is not the opposite halfspace of $\mathfrak{h}_2$, then a hyperplane $\hat{\mathfrak{h}}$ is a horizontal hyperplane of $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ if and only if $\hat{\mathfrak{h}}$ crosses both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$.*

Proof. From Remark 6.3.9, every horizontal hyperplane of $B(\hat{\mathfrak{h}}_1, \hat{\mathfrak{h}}_2)$ is a horizontal hyperplane of $B(\mathfrak{h}_1, \mathfrak{h}_2)$. So we need only to show the statement for $B(\mathfrak{h}_1, \mathfrak{h}_2)$.

If $\mathfrak{h}_1$ is the opposite halfspace of $\mathfrak{h}_2$, then the bridge between $\mathfrak{h}_1$ and $\mathfrak{h}_2$ is equal to the carrier of $\hat{\mathfrak{h}}_1$. So every hyperplane crossing $\hat{\mathfrak{h}}_1$ is a horizontal hyperplane.

Assume that $\mathfrak{h}_1 \neq \mathfrak{h}_2^*$. Choose a hyperplane $\hat{\mathfrak{h}}$ crossing both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$. Let $\mathfrak{h}$ be a halfspace of $\hat{\mathfrak{h}}$. Then, for each $j \in \{1, 2\}$, there is a square S_j where $\hat{\mathfrak{h}}$ crosses $\hat{\mathfrak{h}}_j$. Let x_j and y_j be the intersections $S_j \cap \mathfrak{h}_j \cap \mathfrak{h}$ and $S_j \cap \mathfrak{h}_j \cap \mathfrak{h}^*$, respectively. By Remark 6.3.10, we have

$$\begin{aligned} d(x_1, x_2) &= d(x_1, \pi_B(x_1)) + d(\pi_B(x_1), \pi_B(x_2)) + d(\pi_B(x_2), x_2) \text{ and} \\ d(y_1, y_2) &= d(y_1, \pi_B(y_1)) + d(\pi_B(y_1), \pi_B(y_2)) + d(\pi_B(y_2), y_2) \end{aligned}$$

where B denotes the bridge $B(\mathfrak{h}_1, \mathfrak{h}_2)$. Since $\mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}$ and $\mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}^*$ are convex (by Remark 5.4.14) and $x_j \in \mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}$ for each j, we have $\pi_B(x_1), \pi_B(x_2) \in \mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}$. Similarly, $\pi_B(y_1), \pi_B(y_2) \in \mathcal{N}(\hat{\mathfrak{h}}) \cap \mathfrak{h}^*$. They impliy that $\mathfrak{h} \cap B$ and $\mathfrak{h}^* \cap B$ are nonempty. Hence, $\hat{\mathfrak{h}}$ passes through B by Lemma 5.2.20. $\square$

Chapter 7

Actions of fundamental groups of surfaces on dual cube complexes

We write Σ as a hyperbolic surface of finite area with a covering map $\xi : \mathbb{H}^2 \to \Sigma$. Fix a finite set of simple closed geodesics, denoted by $\mathcal{F}$. We borrow symbols on the symbol list next to the reference page.

7.1 Simple closed geodesics and wallspaces

Lemma 7.1.1. *The dual cube complex of $\mathcal{F}$ is of dimension at most $|\mathcal{F}|$.*

Proof. Let C be an arbitrary n-cube of $\tilde{X}$. Then all hyperplanes $\hat{\mathfrak{h}}_1, \dots, \hat{\mathfrak{h}}_n$ passing through C cross each other. For each $j = 1, \dots, n$, let $\tilde{\gamma}_j$ be a wall in $\tilde{\mathcal{F}}$ such that $\tau(\tilde{\gamma}_j) = \hat{\mathfrak{h}}_j$. Then $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ cross each other. Because $\xi(\tilde{\gamma}_j)$ does not have any self-intersection, the map $j \mapsto \xi(\tilde{\gamma}_j)$ is injective. So $n \leq |\mathcal{F}|$. That is, every cube of $\tilde{X}$ has dimension at most $|\mathcal{F}|$. Therefore, the dimension of $\tilde{X}$ is at most $|\mathcal{F}|$. □

Remark 7.1.2. For every $\gamma \in \mathcal{F}$ and every pair of vertices $u, v \in \tilde{X}_\gamma$, the set $\mathcal{I}(v) - \mathcal{I}(u) = \mathcal{O}(u) \cap \mathcal{I}(v)$ is totally ordered, i.e., $\{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ because there are no transverse halfspaces by Lemma 2.4.8.

Lemma 7.1.3. *For each pair of vertices $x, y \in \tilde{X}$, we have*

$$d(x, y) = \sum_{\gamma \in \mathcal{F}} d_\gamma(\pi_\gamma(x), \pi_\gamma(y)).$$

Proof. Let $\mathcal{I}(x)$ and $\mathcal{I}(y)$ be the inward orientations of x and y. By Remark 5.4.9, $d(x,y) = |\mathcal{I}(x) - \mathcal{I}(y)|$. Because τ is bijective, $|\mathcal{I}(x) - \mathcal{I}(y)| = |\tau^{-1}(\mathcal{I}(x) - \mathcal{I}(y))|$. Since $\mathcal{H}^{\mathbb{H}^2}$ is decomposed into $\bigsqcup_{\gamma\in\mathcal{F}} \mathcal{H}^{\mathbb{H}^2}_\gamma$,

$$\tau^{-1}(\mathcal{I}(x) - \mathcal{I}(y)) = \bigsqcup_{\gamma\in\mathcal{F}} (\mathcal{H}^{\mathbb{H}^2}_\gamma \cap \tau^{-1}(\mathcal{I}(x) - \mathcal{I}(y))).$$

So

$$d(x,y) = \sum_{\gamma\in\mathcal{F}} |\mathcal{H}^{\mathbb{H}^2}_\gamma \cap \tau^{-1}(\mathcal{I}(x) - \mathcal{I}(y))|.$$

By the bijectivity of τ_γ and Proposition 6.1.2,

$$\begin{aligned} d(x,y) &= \sum_{\gamma\in\mathcal{F}} |\tau_\gamma(\mathcal{H}^{\mathbb{H}^2}_\gamma \cap \tau^{-1}(\mathcal{I}(x) - \mathcal{I}(y)))| \\ &= \sum_{\gamma\in\mathcal{F}} |\mathcal{I}(\pi_\gamma(x)) - \mathcal{I}(\pi_\gamma(y))| \\ &= \sum_{\gamma\in\mathcal{F}} d_\gamma(\pi_\gamma(x), \pi_\gamma(y)). \end{aligned}$$

□

If a principal ultrafilter $\mathcal{U}$ contains three halfspaces, then the intersection of all halfspaces in $\mathcal{U}$ is empty. But every principal ultrafilter consisting of halfspaces of $\tilde{X}$ is the inward orientation of some vertex, that is, the intersection of all halfspaces is nonempty.

Lemma 7.1.4. *Assume that $\mathcal{F}$ is a finite set of simple closed geodesics, which satisfies the dimension of the dual cube complex $\tilde{X}$ is at most 2. Then for every vertex x on $\tilde{X}$, there is a point p on $\mathbb{H}^2$ such that the inward orientation of x is equal to $\{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}^{\mathbb{H}^2}, p \in \mathrm{int}(\mathsf{H})\}$.*

Remark 7.1.5. Let us point out the following facts used in the proof of Lemma 7.1.4.

1. The hypothesis in Lemma 7.1.4 implies that there are no triple of halfspaces of $\mathcal{H}^{\mathbb{H}^2}$ which are transverse to each other.

2. If $\mathcal{U}$ is a principal ultrafilter, and if H_1 and H_2 are minimal halfspaces of $\mathcal{U}$, then either $\mathsf{H}_1^* \subsetneq \mathsf{H}_2$ or they are transverse to each other.

3. If ϵ is a positive constant less than the minimum of the collar lengths of simple closed geodesics in $\mathcal{F}$, then the following holds. Let H_1 and H_2 be halfspaces in $\mathcal{H}^{\mathbb{H}^2}$ such that $\mathsf{H}_1^* \subsetneq \mathsf{H}_2$. If $\tilde{\gamma}_1$ is the boundary of H_1, then the ϵ-neighborhood of $\tilde{\gamma}_1$, denoted by $\mathcal{N}_\epsilon(\tilde{\gamma})$, is contained in H_2 as a subset.

Proof of Lemma 7.1.4. Let ϵ be a positive constant which is less than the collar lengths of all simple closed geodesics of $\mathcal{F}$. Let $\mathcal{U}$ be the principal ultrafilter of $\mathcal{H}^{\mathbb{H}^2}$ such that $\mathcal{I}(x) = \{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{U}\}$. Then it is enough to show that the intersection $\bigcap_{\mathsf{H}\in\mathcal{U}} \mathrm{int}(\mathsf{H})$ is nonempty. ($\mathrm{int}(\mathsf{H})$ denotes the interior of H.) Let $\mathcal{U}_m$ denotes the set of all minimal halfspaces of $\mathcal{U}$. Note that $\bigcap_{\mathsf{H}\in\mathcal{U}} \mathrm{int}(\mathsf{H}) = \bigcap_{\mathsf{H}\in\mathcal{U}_m} \mathrm{int}(\mathsf{H})$ since every halfspace of $\mathcal{U}$ contains some minimal halfspace of $\mathcal{U}$ as a subset.

Let **INE** be the collection of subsets of $\mathcal{U}_m$ such that $\mathcal{V}$ belongs to **INE** if and only if $\bigcap_{\mathsf{H}\in\mathcal{V}} \mathrm{int}(\mathsf{H})$ is nonempty. Then **INE** is partially ordered with the set-inclusion. Let $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots$ be an arbitrary chain of **INE**. Choose a halfspace H_1 in $\mathcal{V}_1$. (Note that $\mathcal{V}_1$ is not empty.) If $\tilde{\gamma}_1$ is the boundary of H_1, we choose a point p_1 which is contained in $\mathrm{int}(\mathsf{H}_1) \cap \mathcal{N}_\epsilon(\tilde{\gamma}_1)$. For each $i \in \mathbb{N}$, we choose inductively a halfspace $\mathsf{H} \in \mathcal{V}_i$ and a point p_i satisfying that,

1. if it exists, H_i is transverse to H_j for all $1 \leq j < i$ and p_i lies on $\bigcap_{1\leq j\leq i} (\mathcal{N}_\epsilon(\tilde{\gamma}_j) \cap \mathrm{int}(\mathsf{H}_j))$,
2. otherwise, $\mathsf{H}_i = \mathsf{H}_{i-1}$ and $p_i = p_{i-1}$,

where $\tilde{\gamma}_i$ is the boundary of H_i for each i. Because $\mathcal{F}$ is finite, there is a sufficiently large N such that $\mathsf{H}_i = \mathsf{H}_N$ and $p_i = p_N$ for all $i \geq N$. Let $p := p_N$. Then $p \in \mathcal{N}_\epsilon(\tilde{\gamma}_i) \cap \mathrm{int}(\mathsf{H}_i)$ for all $i \in \mathbb{N}$. Meanwhile, note that there is no halfspace of $\bigcup_{i\in\mathbb{N}} \mathcal{V}_i$ which is transverse to all H_i. So, for every halfspace H of $\bigcup_{i\in\mathbb{N}} \mathcal{V}_i$, there is an integer $i(\mathsf{H})$ such that either $\mathsf{H} = \mathsf{H}_{i(\mathsf{H})}$ or $\mathsf{H}^* \subsetneq \mathsf{H}_{i(\mathsf{H})}$. Both cases imply that $p \in \mathsf{H}$. (When $\mathsf{H}^* \subsetneq \mathsf{H}_{i(\mathsf{H})}$, the halfspace H includes $\mathcal{N}_\epsilon(\tilde{\gamma}_i) \cap \mathrm{int}(\mathsf{H}_i)$ as a subset.) Then $\bigcap_{\mathsf{H}\in\bigcup\mathcal{V}_i} \mathrm{int}(\mathsf{H})$ contains p, which is hence nonempty.

By Zorn's lemma, there is a maximal set $\mathcal{V}$ in the collection **INE**. For contradiction, suppose that $\mathcal{V} \subsetneq \mathcal{U}_m$. Let H_0 be a halfspace in the complement of $\mathcal{V}$, and let $\tilde{\gamma}_0$ be the boundary of H_0. Note that H_0 is disjoint from $\bigcap_{\mathsf{H}\in\mathcal{V}} \mathrm{int}(\mathsf{H})$. Let $\mathcal{V}^t$ denote $\{\mathsf{H} \in \mathcal{V} \,|\, \mathsf{H} \text{ is transverse to } \mathsf{H}_0\}$, and let $\mathcal{V}^p$ be $\{\mathsf{H} \in \mathcal{V} \,|\, \mathsf{H}^* \subsetneq \mathsf{H}_0\}$. Then $\mathcal{V}$ is the disjoint union of $\mathcal{V}^t$ and $\mathcal{V}^p$. If $\mathcal{V}^t$ is empty, then $\mathcal{N}_\epsilon(\tilde{\gamma}_0) \cap \mathrm{int}(\mathsf{H}_0) \in \mathrm{int}(\mathsf{H})$ for all $\mathsf{H} \in \mathcal{V}^p = \mathcal{V}$. So $\{\mathsf{H}_0\} \cup \mathcal{V} \in$ **INE**, which is a contradiction. It induces that there is a halfspace H' in $\mathcal{V}^t$.

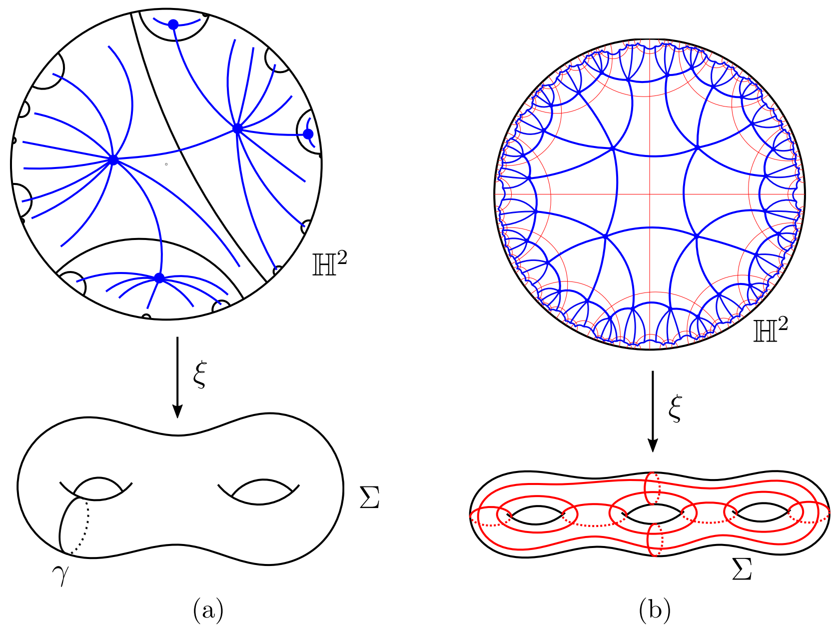

Figure 7.1: If $\mathcal{F}$ satisfies the hypothesis of Lemma 7.1.4, then there is a canonical way to embed $\tilde{X}$ into $\mathbb{H}^2$. The blue graphs are the parts of the embeddings.

By the hypothesis, other hyperplanes of $\mathcal{V}^t$ cannot be transverse to H'. If $\tilde{\gamma}'$ is the boundary of H', then $(\mathcal{N}_\epsilon(\tilde{\gamma}_0) \cap \operatorname{int}(\mathsf{H}_0)) \cap (\mathcal{N}_\epsilon(\tilde{\gamma}') \cap \operatorname{int}(\mathsf{H}'))$ is nonempty and is contained in the interiors of all halfspaces of $\mathcal{V}$. Then it also implies that $\{\mathsf{H}_0\} \cup \mathcal{V}$ belongs to **INE**, whcih is a contradiction against the maximality of $\mathcal{V}$. Therefore, $\mathcal{V} = \mathcal{U}_m$, that is, $\bigcap_{\mathsf{H}\in\mathcal{U}_m} \mathsf{H}$ is nonempty. $\square$

Example 5. If $\mathcal{F}$ satisfies the hypothesis of Lemma 7.1.4, then there is a way to embed $\tilde{X}$ into $\mathbb{H}^2$. For each vertex $x \in \tilde{X}$, let p_x be a point of $\mathbb{H}^2$ such that

$$\mathcal{I}(x) = \{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}^{\mathbb{H}^2}, p \in \mathsf{H}\}.$$

Whenever two vertices x and y are joined by an edge, let us draw the geodesic segment joining p_x and p_y. If $\mathcal{F}$ contains exactly one simple closed geodesic γ, then $\tilde{X}$ is a tree by Lemma 7.1.3. The blue object in Figure 7.1a is a part of the embedded tree.

Example 6. If Σ is closed and $\mathcal{F}$ fills Σ, then $\tilde{X}$ homeomoprhism to $\mathbb{H}^2$. See Figure 7.1b. The red curves in Σ represent simple closed geodesics filling a surface of genus 3. Then walls are showed as red lines in $\mathbb{H}^2$. The blue is the 1-skeleton of the embedding of $\tilde{X}$, constructed in Example 5. In this case, the intersection point of every pair of walls are contained in some square of the embedded complex.

We can generalize the above examples to the fact that, if $\mathcal{F}$ satisfies the hypothesis of Lemma 7.1.4, then we can construct an embedding of $\tilde{X}$ into $\mathbb{H}^2$, but we do not prove it. Since both $\mathbb{H}^2$ and $\tilde{X}$ are contractible, the embedding of $\tilde{X}$ into $\mathbb{H}^2$ is considered as a homotopy equivalence. That is, we can find a continuous map from $\mathbb{H}^2$ to $\tilde{X}$, which is an inverse of the embedding in the sense of homotopy equivalence. In fact, this map can be surjective and $\pi_1(\Sigma)$-equivariant. The following proposition is what we need.

Proposition 7.1.6. *The action of $\pi_1(\Sigma)$ on the dual cube complex $\tilde{X}$ is cocompact.*

Remark 7.1.7. When a surface Σ is closed, Sageev in [36, Section 4.1 in Lecture 2] proved Proposition 7.1.6. Furthermore, he proved that the action is free whenever Σ is closed and $\mathcal{F}$ fills Σ, (i.e., whenever the complement of $\bigcup_{\gamma\in\mathcal{F}}\gamma$ is the disjoint union of open disks.)

Lemma 7.1.8 (Exercise 2.15, [36])**.** *If geodesics $\tilde{\alpha}_1,\dots,\tilde{\alpha}_n$ in $\mathbb{H}^2$ cross each other for $n\geq 2$, then the number of lifts of $\mathcal{F}$ which intersect all $\tilde{\alpha}_j$ transversally is finite.*

Proof of Lemma 7.1.8. Let θ be the minimum among the angles between $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$. Fix $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ and let p be the intersection point of $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$. Assume that a wall $\tilde{\gamma}$ crosses both $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ but it does not passes through the point p. Let H be the halfspace of $\tilde{\gamma}$ which does not contains p, and let $q:=\tilde{\alpha}_i\cap\tilde{\gamma}$ and $q':=\tilde{\alpha}_j\cap\tilde{\gamma}$. If q_∞ (*resp.*, q'_∞) is the endpoint of the ray $\tilde{\alpha}_i\cap\mathsf{H}$ (*resp.*, $\tilde{\alpha}_j\cap\mathsf{H}$) in the visual boundary $\partial\mathbb{H}^2$, then the geodesic $\tilde{\beta}$ joining q_∞ and q'_∞ is farther from p than $\tilde{\gamma}$ since $\tilde{\beta}$ is contained in H. If p_1 is the point on $\tilde{\beta}$ which is closest to p, then, by the second law of cosine in [32, Theorem 3.5.4],

$$d_{\mathbb{H}^2}(p,\tilde{\gamma}) < d_{\mathbb{H}^2}(p,\tilde{\beta}) = d_{\mathbb{H}^2}(p,p_1) \leq \cosh^{-1}\csc\frac{\theta}{2}.$$

where $d_{\mathbb{H}^2}$ is the metric on $\mathbb{H}^2$. That is, every wall which crosses both $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ intersects the open ball of radius $\cosh^{-1}\csc\frac{\theta}{2}$ with the center p. So if a wall crosses all $\tilde{\alpha}_1,\dots,\tilde{\alpha}_n$, then it must pass through some bounded set B. By the collar lemma, the number of walls which intersect B is finite. $\square$

Proof of Proposition 7.1.6. We follow the proof in [36, Section 4.1 in Lecture 2]. For each $n \geq 1$, let $\mathcal{T}\tilde{\mathcal{F}}_n$ be the collection of subsets of $\tilde{\mathcal{F}}$ consisting of n pairwise transverse walls. We give an element-wise action of $\pi_1(\Sigma)$ on $\mathcal{T}\tilde{\mathcal{F}}_n$, defined by $g\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\} = \{g\tilde{\alpha}_1, \ldots, g\tilde{\alpha}_n\}$ for all $g \in \pi_1(\Sigma)$ and $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\} \in \mathcal{T}\tilde{\mathcal{F}}_n$.

We claim that $\mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ has only finitely many classes for all n. If $n = 1$, then $\mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ is equal to $\mathcal{F}$, so it is finite. For the induction step, assume that $\mathcal{T}\tilde{\mathcal{F}}_{n-1}/\pi_1(\Sigma)$ is finite. Let $2^{\mathcal{F}}$ be the collection of all subsets of $\mathcal{F}$. Define a map $\Phi_n : \mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma) \to 2^{\mathcal{F}}$ by $\Phi_n(\pi_1(\Sigma)\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}) = \{\pi_1(\Sigma)\tilde{\alpha}_1, \ldots, \pi_1(\Sigma)\tilde{\alpha}_n\}$ for each n. Then the image of $\mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ is the set of n pairwise transverse simple closed geodesics in $\mathcal{F}$. Suppose that $\mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ is infinite. Since $2^{\mathcal{F}}$ is finite, there is $\mathcal{F}' \in 2^{\mathcal{F}}$ such that $\Phi_n^{-1}(\mathcal{F}')$ contains infinitely many classes. Choose $\gamma \in \mathcal{F}'$ and consider the forgetful map $\Psi : \Phi_n^{-1}(\mathcal{F}') \to \Phi_{n-1}^{-1}(\mathcal{F}' \setminus \{\gamma\})$. By the induction hypothesis, $\Phi_{n-1}^{-1}(\mathcal{F}' \setminus \{\gamma\})$ contains finitely many classes so that there is $A \in \Phi_{n-1}^{-1}(\mathcal{F}' \setminus \{\gamma\})$ such that $\Psi^{-1}(A)$ has infinitely many classes. It implies that there are infinitely many lifts of γ, each of which intersects all geodesics in A. But it is a contradiction by Lemma 7.1.8. So $\mathcal{T}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ contains finitely many classes.

For each $n \geq 1$, let $\mathcal{MT}\tilde{\mathcal{F}}_n$ denote $\{A \in \mathcal{T}\tilde{\mathcal{F}}_n \,|\, \text{there is no wall of } \tilde{\mathcal{F}} \text{ which crosses all walls of } A\}$. Then $\mathcal{MT}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)$ is also finite for each n. Since $\tilde{\mathcal{F}}$ does not contain more than $|\mathcal{F}|$ pairwise transverse walls, the union $\bigcup_{n\geq 1}(\mathcal{MT}\tilde{\mathcal{F}}_n/\pi_1(\Sigma))$ is also finite. Let $A_1, \ldots, A_m \subset \tilde{\mathcal{F}}$ be representatives such that $\bigcup_{n\geq 1}(\mathcal{MT}\tilde{\mathcal{F}}_n/\pi_1(\Sigma)) = \{\pi_1(\Sigma)A_j \,|\, j = 1, \ldots, m\}$. For each $j = 1, \ldots, m$, the intersection $\bigcap_{\tilde{\gamma}\in A_j} \tau(\tilde{\gamma})$ is nonempty by Proposition 5.2.15.(3). And, by maximality, the intersection is a point. For each j, let C_j be the cube which contains $\bigcap_{\tilde{\gamma}\in A_j} \tau(\tilde{\gamma})$.

We claim that, for every cube C, there are $g \in \pi_1(\Sigma)$ and $j \in \{1, \ldots, m\}$ such that $gC \subset C_j$. Let C' be the maximal cube which contains C. (That is, there is no cube which contains C' properly.) Then the hyperplanes $\hat{\mathfrak{h}}_1, \ldots, \hat{\mathfrak{h}}_n$ passing through C' are transverse to each other. So $\tau^{-1}(\hat{\mathfrak{h}}_1), \ldots, \tau^{-1}(\hat{\mathfrak{h}}_n)$ cross pairwise. Because C' is maximal, the set $A := \{\tau^{-1}(\hat{\mathfrak{h}}_1), \ldots, \tau^{-1}(\hat{\mathfrak{h}}_n)\}$ belongs to the collection $\mathcal{MT}\tilde{\mathcal{F}}_n$. Then there are $g \in \pi_1(\Sigma)$ and $j \in \{1, \ldots, m\}$ such that $gA = A_j$. It implies that $gC' = C_j$ and that $gC \subset C_j$.

Therefore, $\tilde{X}$ is covered by gC_j for all $g \in \pi_1(\Sigma)$ and $j \in \{1, \ldots, m\}$. □

If more than one wall of $\tilde{\mathcal{F}}$ are preserved by some $\pi_1(\Sigma)$ of $\pi_1(\Sigma)$, then $\pi_1(\Sigma)$ is the identity map. It implies that every nonidentity element of $\pi_1(\Sigma)$ moves some hyperplane of $\tilde{X}$.

Lemma 7.1.9. *The action of* $\pi_1(\Sigma)$ *on* $\tilde{X}$ *is faithful.*

Because $\mathcal{F}$ is arbitrary, it holds that $\pi_1(\Sigma)$ acts on $\tilde{X}_\gamma$ faithfully for each $\gamma \in \mathcal{F}$.

Remark 7.1.10. For every $\gamma \in \mathcal{F}$, if $\tilde{\gamma}$ and $\tilde{\gamma}'$ are walls of $\tilde{\mathcal{F}}_\gamma$, then we have $\tilde{\gamma}' = g\tilde{\gamma}$ for some $g \in \pi_1(\Sigma)$. So $\pi_1(\Sigma)$ acts transitively on the set of edges of $\tilde{X}_\gamma$.

Example 7. Although the orbit of an edge covers $\tilde{X}_\gamma$, it is not true, in general, that the orbit of a vertex covers the 0-skeleton of $\tilde{X}$. Suppose that $\gamma \in \mathcal{F}$ is a separating simple closed geodesic, i.e., $\Sigma \setminus \gamma$ is disconnected. Whenever a closed geodesic of Σ intersects γ, it must intersect γ more than once. Choose two vertices u and v of $\tilde{X}_\gamma$ which are joined by an edge. If p and q are points of $\mathbb{H}^2$ such that $\mathcal{I}(u) = \{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2}, p \in \mathsf{H}\}$ and $\mathcal{I}(v) = \{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2}, q \in \mathsf{H}\}$, then the geodesic joining p and q intersects only one wall of $\tilde{\mathcal{F}}_\gamma$. So there is no element of $\pi_1(\Sigma)$ which sends p to q.

Proposition 7.1.11. *A simple closed geodesic* γ *is non-separating if and only if the action of* $\pi_1(\Sigma)$ *on the* 0*-skeleton of* $\tilde{X}_\gamma$ *is transitive.*

Proof. Suppose that there is an oriented edge $\vec{e}$ which joins u to v. Let H_0 be the halfspace of $(\mathbb{H}^2, \tilde{\mathcal{F}}_\gamma)$ such that $\tau_\gamma(\mathsf{H}_0)$ is the terminal halfspace of $\vec{e}$. And let $\tilde{\gamma}_0$ be the wall which bounds H_0. Let ϵ be a positive number satisfying that the closed ϵ-neighborhood of γ is homeomorphic to the cylinder $S^1 \times [0, 1]$.

Since γ is non-separating, there is a simple closed geodesic α such that $|\gamma \cap \alpha| = 1$. Let $\tilde{\alpha}$ be a lift of α which intersects $\tilde{\gamma}_0$. Choose a point p in $(\tilde{\alpha} \cap \mathcal{N}_\epsilon(\tilde{\gamma}_0) \cap \mathsf{H}_0^*) - \tilde{\gamma}_0$. Then there is a primitive element g_0 of $\pi_1(\Sigma)$ such that $\tilde{\alpha}$ is the axis of g_0 and it separates p from g_0p. The geodesic joining p and g_0p is a subarc of $\tilde{\alpha}$ and it does not intersect any walls in $\tilde{\mathcal{F}}_\gamma$ other than $\tilde{\gamma}_0$.

Let

$$\mathcal{I}(p) := \{\mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2} \,|\, p \in \mathsf{H}\} \text{ and}$$
$$\mathcal{I}(g_0p) := \{\mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2} \,|\, g_0p \in \mathsf{H}\}.$$

Then we can easily verfy that $\mathcal{I}(p)$ and $\mathcal{I}(g_0p)$ are principal ultrafilters.

We claim that the inward orientation $\mathcal{I}(u)$ of u is equal to $\tau_\gamma\mathcal{I}(p) := \{\tau_\gamma(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{I}(p)\}$ and that $\mathcal{I}(v)$ is equal to $\tau_\gamma\mathcal{I}(g_0p) := \{\tau_\gamma(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{I}(g_0p)\}$.

First, note that $\tau_\gamma\mathcal{I}(p)$ contains $\tau_\gamma(\mathsf{H}_0^*)$ as a minimal element. So $(\tau_\gamma\mathcal{I}(p) \setminus \{\tau_\gamma(\mathsf{H}_0^*)\}) \cup \{\tau_\gamma(\mathsf{H}_0)\}$ is also a principal ultrafilter. Then the vertex whose inward orientation is $\tau_\gamma\mathcal{I}(p)$ is contained in the carrier of $\tau_\gamma(\tilde{\gamma}_0)$. Because $\tilde{X}_\gamma$ is a tree by Lemma 7.1.1, the carrier of $\tau_\gamma(\tilde{\gamma}_0)$ is exactly the edge $\vec{e}$, so $\tau_\gamma\mathcal{I}(p)$ is the inward orientation of u. Since $\tilde{\gamma}$ is the unique wall of $\tilde{\mathcal{F}}_\gamma$ which separates p from g_0p, we have $\tau_\gamma\mathcal{I}(g_0p) - \tau_\gamma\mathcal{I}(p) = \{\tau_\gamma(\mathsf{H}_0)\}$. So the vertex whose inward orientation is $\tau_\gamma\mathcal{I}(g_0p)$ is joined to u by the edge $\vec{e}$. That is, it is v.

By the above claim,

$$\begin{aligned}
\mathcal{I}(g_0u) &= \{\mathfrak{h} \in \mathcal{H}_\gamma \,|\, g_0u \in \mathfrak{h}\} \\
&= \{g_0\mathfrak{h} \,|\, \mathfrak{h} \in \mathcal{H}_\gamma, u \in \mathfrak{h}\} = \{g_0\mathfrak{h} \,|\, \mathfrak{h} \in \mathcal{I}(u)\} \\
&= \{g_0\tau_\gamma(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{I}(p)\} = \{g_0\tau_\gamma(\mathsf{H}) \,|\, p \in \mathsf{H}\} \\
&= \{\tau_\gamma(\mathsf{H}) \,|\, g_0p \in \mathsf{H}\} = \tau_\gamma\mathcal{I}(g_0p) \\
&= \mathcal{I}(v).
\end{aligned}$$

Then we have $v = g_0u$.

Let u and v be arbitrary vertices of $\tilde{X}_\gamma$. Let $\vec{L} = (\vec{e}_j)_{j=1,\dots,m}$ be the geodesic path from u to v and v_j the terminal vertex of $\vec{e}_j$ for each j. If v_0 denotes the vertex u, then, for each $j = 1, \dots, m$, there is an element $g_j \in \pi_1(\Sigma)$ such that $v_j = g_jv_{j-1}$. Therefore, we have $v = v_m = g_mg_{m-1}\dots g_2g_1u$. □

7.2 Hyperbolic isometries on dual cube complexes

Lemma 7.2.1. *If a nonidentity element g of $\pi_1(\Sigma)$ is a parabolic isometry on $\mathbb{H}^2$, then g fixes a vertex of $\tilde{X}$ but not any hyperplane.*

Proof. Because g is parabolic on $\mathbb{H}^2$, there is no geodesic of $\tilde{\mathcal{F}}$ which is invariant by g. So g does not fix any hyperplane. Let q_∞ be an ideal point of $\mathbb{H}^2$ which is fixed by g. Because q_∞ is a cusp in Σ, there is a horodisk $B(q_\infty)$ centered at q_∞ such that any wall in L does not intersect with $B(q_\infty)$. (Refer to [32, Theorem 9.8.4].) So the set $\mathcal{I} := \{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}^{\mathbb{H}^2}, B(q_\infty) \subset \mathsf{H}\}$ is a principal ultrafilter. Hence, g fixes the vertex whose inward orientation is $\mathcal{I}$. □

Definition 7.2.2 (Translation length revisited)**.** For each $g \in \pi_1(\Sigma)$, the *translation length* of g (with respect to the combinatorial metric of $\tilde{X}$) is defined by $\inf_{x \in \tilde{X}^{(0)}} d(x, gx)$ and denoted by $\operatorname{tr} g$ or $\operatorname{tr}_{\mathcal{F}} g$. And, for each $\gamma \in \mathcal{F}$, the *translation length* of g with respect to the combinatorial metric of $\tilde{X}_\gamma$ is denoted by $\operatorname{tr}_\gamma g$.

Lemma 7.2.1 says that, if g in $\pi_1(\Sigma)$ is a parabolic element in $\mathbb{H}^2$, then $\operatorname{tr}_{\mathcal{F}} g = 0$.

Definition 7.2.3. For each $g \in \pi_1(\Sigma)$, let $\mathcal{GH}(g, \mathcal{F})$ denote the set of halfspaces $\mathfrak{h}$ of $\tilde{X}$ satisfying that $g\mathfrak{h} \subsetneq \mathfrak{h}$. If $\mathcal{F}$ is specified, then we simply write $\mathcal{GH}(g)$.

Remark 7.2.4. Let $g \in \pi_1(\Sigma)$ be given.

- If $\mathfrak{h}$ is a halfspace of $\mathcal{GH}(g)$, then the opposite halfspace $\mathfrak{h}^*$ is not an element of $\mathcal{GH}(g)$ because $g\mathfrak{h}^* \supsetneq \mathfrak{h}^*$.
- For every halfspace $\mathfrak{h} \in \mathcal{GH}(g)$ and an integer n, we have $g(g^n\mathfrak{h}) = g^n(g\mathfrak{h}) \subsetneq g^n\mathfrak{h}$. That is, $g^n\mathfrak{h} \in \mathcal{GH}(g)$.
- For every $\mathfrak{h} \in \mathcal{GH}(g)$, the intersection $\bigcap_{n \in \mathbb{Z}_+} g^n\mathfrak{h}$ is empty.

Lemma 7.2.5. *Let an element g of $\pi_1(\Sigma)$ be given. If two distinct halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are contained in $\mathcal{GH}(g)$, then one of the following happens.*

1. $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2$,
2. $\mathfrak{h}_2 \supsetneq \mathfrak{h}_2$, *or*
3. $\mathfrak{h}_1$ *is transverse to* $\mathfrak{h}_2$.

Proof. Assume not. Then either $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2^*$ or $\mathfrak{h}_2^* \supsetneq \mathfrak{h}_1$. If $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2^*$, then

$$g^n\mathfrak{h}_1 \supsetneq g^n\mathfrak{h}_2^* \supsetneq \mathfrak{h}_2^*$$

for every positive integer n. It implies that the intersection $\bigcap_{n>0} g^n\mathfrak{h}_1$ is nonempty, which is a contradiction. So it cannot happen that $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2^*$. Similarly, the case that $\mathfrak{h}_1^* \supsetneq \mathfrak{h}_2$ is not able to occur. $\square$

Lemma 7.2.6. *Let $g \in \pi_1(\Sigma)$ be given.*

1. *If $\mathcal{GH}(g)$ is nonempty, then g is a hyperbolic isometry on $\mathbb{H}^2$.*

2. *A halfspace* $\mathfrak{h}$ *or its opposite halfspace* $\mathfrak{h}^*$ *is contained in* $\mathcal{GH}(g)$ *if and only if* g *is hyperbolic on* $\mathbb{H}^2$ *and the axis of* g *crosses the geodesic* $\tau^{-1}(\hat{\mathfrak{h}})$*.*

3. *The set* $\mathcal{GH}(g)$ *is nonempty if and only if the translation length of* g *in* $\tilde{X}$ *is nonzero.*

Proof. (1) Choose a halfspace $\mathfrak{h}$ in $\mathcal{GH}(g)$. If $\tilde{\gamma}$ is the wall bounding $\mathsf{H} := \tau^{-1}(\mathfrak{h})$, then, by the collar lemma, $d_{\mathbb{H}^2}(\tilde{\gamma}, g\tilde{\gamma}) \geq R$ for some $R > 0$. Choose a point q in $\mathbb{H}^2$. Because $\mathbb{H}^2 = \bigcup_{n\in\mathbb{Z}} g^n\mathsf{H}$, there is an integer $n \in \mathbb{Z}$ such that q is contained in $g^n\mathsf{H} \cap g^{n+1}\mathsf{H}^*$. Since $g^2q \in g^{n+2}\mathsf{H}$ and $\mathsf{H} \supsetneq g\mathsf{H}$, we have

$$\begin{aligned} d_{\mathbb{H}^2}(q, g^2q) &\geq d_{\mathbb{H}^2}(g^{n+1}\mathsf{H}^*, g^{n+2}\mathsf{H}) \\ &= d_{\mathbb{H}^2}(\mathsf{H}^*, g\mathsf{H}) \\ &\geq R \end{aligned}$$

Because a point q in $\mathbb{H}^2$ is arbitrarily chosen, g^2 is hyperbolic, and so is g.

(2) Let $\tilde{\gamma}_g$ be the axis of g. Assume that a wall $\tilde{\gamma}$ in $\tilde{\mathcal{F}}$ intersects $\tilde{\gamma}_g$ transversally. Since $\tilde{\gamma}$ and $g\tilde{\gamma}$ are lifts of a simple closed geodesic, they are disjoint from each other. Then either a halfspace H of $\tilde{\gamma}$ or its opposite halfspace H^* contains $g\tilde{\gamma}$. Without loss of generality, we suppose that $g\tilde{\gamma} \subset \mathsf{H}$. If $g\mathsf{H}$ is not a proper subset of H, then $\tilde{\gamma}_g \cap \mathsf{H} \cap g\mathsf{H}$ is a compact set so that there is a point fixed by g. It gives a contradiction. So $\mathsf{H} \supsetneq g\mathsf{H}$. Therefore, $\tau(\mathsf{H}) \supsetneq g\tau(\mathsf{H})$.

Conversely, assume that $\mathfrak{h}$ is a halfspace in $\mathcal{GH}(g)$. Let H denote $\tau^{-1}(\mathfrak{h})$ and $\tilde{\gamma}$ the wall of H. Then $\xi(\tilde{\gamma}) \in \mathcal{F}$. Let R be a positive number such that the closed R-neighborhood of $\xi(\tilde{\gamma})$ is homeomorphic to an annulus. Since $\tilde{\gamma}$ and $g\tilde{\gamma}$ are disjoint from each other, the (minimal) distance between them is at least R. For each positive integer n, because $\mathsf{H} \supsetneq g\mathsf{H} \supsetneq \cdots \supsetneq g^n\mathsf{H}$, the distance between $\tilde{\gamma}$ and $g^n\tilde{\gamma}$ is equal to or more than nR. So each point in $\mathbb{H}^2$ locates from the outside of some $g^n\mathsf{H}$. That is, the intersection of all $g^n\mathsf{H}$ is empty.

If p is a point on $\tilde{\gamma}_g$, then there is an integer n such that $p \in g^n\mathsf{H}$ and $p \notin g^{n+1}\mathsf{H}$. Since gp lies on $g^{n+1}\mathsf{H}$ and $p \in g^{n+1}\mathsf{H}^*$, the axis $\tilde{\gamma}_g$ crosses $g^{n+1}\tilde{\gamma}$. Hence, $\tilde{\gamma}_g$ also crosses $\tilde{\gamma}$.

(3) Assume that $\mathcal{GH}(g)$ is empty. If g is a hyperbolic isometry in $\mathbb{H}^2$, then any wall in $\tilde{\mathcal{F}}$ does not cross the axis of g. If $\tilde{\gamma}_g$ is the axis of g, then $\{\mathfrak{h} \in \mathcal{H} \mid \tilde{\gamma}_g \subset \tau^{-1}(\mathfrak{h})\}$ is a principal ultrafilter which is invariant by g. So $\operatorname{tr} g = 0$.

Conversely, suppose that there is a halfspace $\mathfrak{h} \in \mathcal{GH}(g)$. Since the sequence $\cdots \supsetneq g^{-1}\mathfrak{h} \supsetneq \mathfrak{h} \supsetneq g\mathfrak{h} \supsetneq \cdots$ have neither a maximal halfspace nor a minimal halfspace, they cover $\tilde{X}$ and their intersection is empty. Then, for each vertex x of $\tilde{X}$, there is an integer n such that $x \in g^n\mathfrak{h}$ and $x \in g^{n+1}\mathfrak{h}^*$. So $d(x, gx) \geq d(g^{n+1}\mathfrak{h}^*, g^{n+1}\mathfrak{h}) = d(\mathfrak{h}^*, \mathfrak{h}) = 1$. Hence, g does not fix any vertex x in $\tilde{X}$, which means the translation length of g is nonzero. □

Proposition 7.2.7. *The translation length of g on $\tilde{X}$ is equal to the number of $\langle g\rangle$-orbits in $\mathcal{GH}(g)$, that is,* $\operatorname{tr} g = |\mathcal{GH}(g)/\langle g\rangle|$.

Proof. By Lemma 7.2.6.(3), the translation length of g is zero if and only if $|\langle g\rangle\backslash\mathcal{GH}(g)| = 0$. Assume that $\mathcal{GH}(g)$ is nonempty. We will use an induction on $k := |\mathcal{F}|$.

If $k = 1$, then $\tilde{X}$ is a tree by Lemma 7.1.1. Since $\operatorname{tr} g > 0$ by Lemma 7.2.6.(3), there is a unique combinatorial geodesic L in $\tilde{X}$ which is invariant from g by Bass-Serre theory, (precisely, by [38, Proposition 24 in Chapter I].) Let $\vec{L} = (\vec{e}_n)_{n\in\mathbb{Z}}$ be a geodesic path whose image is L and satisfying that $n < m$ whenever $\vec{e}_m = g\vec{e}_n$.

We claim that a halfspace $\mathfrak{h}$ belongs to $\mathcal{GH}(g)$ if and only if $\mathfrak{h}$ is the terminal halfspace of some $\vec{e} \in \vec{L}$. Suppose that a halfspace $\mathfrak{h}$ is the terminal halfspace of an oriented edge $\vec{e} \in \vec{L}$. Since $\tilde{X}$ is a tree and $g\vec{e} \subset \mathfrak{h}$, Lemma 5.4.10 indicates that $g\mathfrak{h} \subsetneq \mathfrak{h}$. So we have $\mathfrak{h} \in \mathcal{GH}(g)$.

Conversely, assume that the oriented edge $\vec{e}$ whose terminal halfspace is $\mathfrak{h}$ does not belong to $\vec{L}$. If $\vec{L}$ has the edge which is oriented backward from $\vec{e}$, then the opposite halfspace of $\mathfrak{h}$ belongs to $\mathcal{GH}(g)$ so that $\mathfrak{h} \subsetneq g\mathfrak{h}$. That is, $\mathfrak{h}$ is not contained in $\mathcal{GH}(g)$. Suppose that $\vec{e}$ does not lie on L. Then either $\mathfrak{h}$ contains L or it is disjoint from L. If L is a subset of $\mathfrak{h}$, then $g^n\mathfrak{h}$ contains L for every positive integer n. So it cannot happen that $g\mathfrak{h} \subsetneq \mathfrak{h}$ by Remark 7.2.4. Finally, if L is disjoint from $\mathfrak{h}$, then $g\mathfrak{h}$ is also disjoint from L. Let x be the vertex on L which is closest to $\mathfrak{h}$. Note that every path from $\mathfrak{h}$ to $g\mathfrak{h}$ must contains all edges of the geodesic $[x, gx]$ joining x and gx since $\tilde{X}$ is a acyclic. Because $[x, gx]$ is a subset of L, it is disjoint from $\mathfrak{h}$ and the complement of $[x, gx]$ contains $\mathfrak{h}$ and $g\mathfrak{h}$ in different connected components. That is, $\mathfrak{h}$ is disjoint from $g\mathfrak{h}$, and then $\mathfrak{h}$ does not belong to $\mathcal{GH}(g)$. So we proved the claim.

Given a vertex x in L, we have $\operatorname{tr} g = d(x, gx)$ by [38, Proposition 24 in Chapter I]. If $\mathcal{I}(gx) \cap \mathcal{O}(x) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_m\}$, then $\mathcal{GH}(g)$ is the disjoint union of $\langle g\rangle$-orbits of $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$. (For more details, for every $\mathfrak{h} \in \mathcal{GH}(g)$, there is an integer n such that $g^n x \in \mathfrak{h}^*$ and $g^{n+1}x \in \mathfrak{h}$, that is, we have $\mathfrak{h} \in \mathcal{I}(g^{n+1}x) \cap \mathcal{O}(g^n x)$. So $g^{-n}\mathfrak{h} \in \mathcal{I}(gx) \cap \mathcal{O}(x)$.) So $\operatorname{tr} g = d(x, gx) = |\{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_m\}| = |\mathcal{GH}(g)/\langle g\rangle|$.

Assume that $k > 1$ Then, by the induction hypothesis and Lemma 7.1.3,

$$\begin{aligned}
\operatorname{tr} g &= \inf_{x \in \tilde{X}^{(0)}} d(x, gx) \\
&\geq \sum_{\gamma \in \mathcal{F}} \inf_{v \in \tilde{X}_\gamma^{(0)}} d(v, gv) = \sum_{\gamma \in \mathcal{F}} \operatorname{tr}_\gamma g \\
&= \sum_{\gamma \in \mathcal{F}} |(\mathcal{GH}(g) \cap \mathcal{H}_\lambda)/\langle g\rangle| \\
&= |\mathcal{GH}(g)/\langle g\rangle|.
\end{aligned}$$

The reason the last equality holds is that $\mathcal{H}$ is partitioned into $\bigcup_{\gamma \in \mathcal{F}} \mathcal{H}_\gamma$. So it is enough to show that $\operatorname{tr} g \leq |\mathcal{GH}(g)/\langle g\rangle|$.

Let $\tilde{\gamma}_g$ be the axis of g in $\mathbb{H}^2$ and choose a point p on $\tilde{\gamma}_g$ such that any wall crossing $\tilde{\gamma}_g$ does not contain p. If $\tilde{\gamma}_g$ is not a wall of $\tilde{\mathcal{F}}$, the set $\mathcal{I}(p) := \{\mathsf{H} \in \mathcal{H}^{\mathbb{H}^2} \mid p \in \mathsf{H}\}$ is a principal ultrafilter of $\mathcal{H}^{\mathbb{H}^2}$, otherwise, $\mathcal{I}(p) \cup \{\mathsf{H}_g\}$ is a principal ultrafilter for a halfspace H_g bounded by $\tilde{\gamma}_g$. Let x be the vertex whose inward orientation is $\tau(\mathcal{I}(p))$ or $\tau(\mathcal{I}(p) \cup \{\mathsf{H}_g\})$. Then the inward orientation of gx is either $\tau(g\mathcal{I}(p))$ or $\tau(g\mathcal{I}(p) \cup \{\mathsf{H}_g\})$. So $d(x, gx) = |g\mathcal{I}(p) - \mathcal{I}(p)|$ by Remark 5.4.9.

Because $g\mathcal{I}(p) - \mathcal{I}(p) = \{\mathsf{H} \in \mathcal{H}^{\mathbb{H}^2} \mid p \in \mathsf{H}^*, gp \in \mathsf{H}\}$, we have $d(x, gx)$ is equal to the number of walls separating p from gp. Note that every wall which separates p from gp crosses $\tilde{\gamma}_g$. By Lemma 7.2.6.(2), we have $\tau(\mathsf{H}) \in \mathcal{GH}(g)$ for all $\mathsf{H} \in g\mathcal{I}(p) - \mathcal{I}(p)$. For each $\mathsf{H} \in g\mathcal{I}(p) - \mathcal{I}(p)$ and $n \in \mathbb{Z}$, we have $g^n\mathsf{H} \in g^{n+1}\mathcal{I}(p) - g^n\mathcal{I}(p)$. So $g\mathcal{I}(p) - \mathcal{I}(p)$ is disjoint from $g^{n+1}\mathcal{I}(p) - g^n\mathcal{I}(p)$ for all nonzero n. Then, whenever $\mathfrak{h}, \mathfrak{h}' \in g\mathcal{I}(p) - \mathcal{I}(p)$ and $\mathfrak{h}' = g^n\mathfrak{h}$, we have $\mathfrak{h} = \mathfrak{h}'$. In other words, the $\langle g\rangle$-orbits of all halfspaces of $g\mathcal{I}(p) - \mathcal{I}(p)$ are different from each other. Totally, $d(x, gx) = |g\mathcal{I}(p) - \mathcal{I}(p)| \leq |\mathcal{GH}(g)/\langle g\rangle|$. □

Lemma 7.2.8. *If $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2 \supsetneq \mathfrak{h}_3$ are halfspaces of $\tilde{X}$ and $\mathfrak{h}_1, \mathfrak{h}_3 \in \mathcal{GH}(g)$, then we have $\mathfrak{h}_2 \in \mathcal{GH}(g)$.*

Proof. Let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ be the walls of $\tau^{-1}(\mathfrak{h}_1)$, $\tau^{-1}(\mathfrak{h}_2)$ and $\tau^{-1}(\mathfrak{h}_3)$, respectively. By the hypothesis, $\tilde{\gamma}_2$ separates $\tilde{\gamma}_1$ from $\tilde{\gamma}_2$. Since both $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ cross the axis of g by Lemma 7.2.6.(2), the wall $\tilde{\gamma}_2$ also intersects transversally the axis of g. Hence, $\mathfrak{h}_2 \in \mathcal{GH}(g)$. □

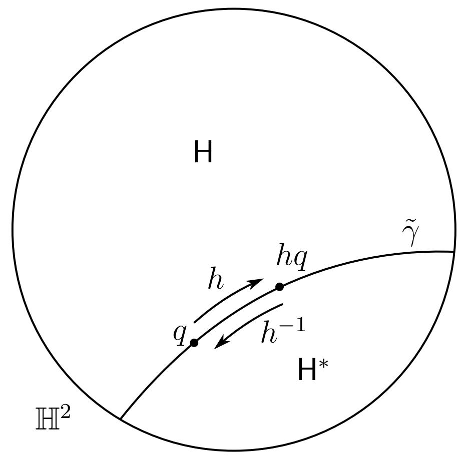

Figure 7.2: $(\tau(\mathsf{H}), h)$ and $(\tau(\mathsf{H}^*), h^{-1})$ are right-handed pairs.

7.3 Right-handed pairs

Definition 7.3.1. For each halfspace H of $\mathbb{H}^2$, the stabilizer of H of $\pi_1(\Sigma)$ is an infinite cyclic subgroup. If $\tilde{\gamma}$ is the boundary of H, then the orientation of H determines an orientation of $\tilde{\gamma}$. Let h be the generator of the stabilzer of H such that, for arbitrary point $p \in \tilde{\gamma}$, the geodesic path from p to hp follows the orientation of $\tilde{\gamma}$. See Figure 7.2. Then we call the pair $(\tau(\mathsf{H}), h) \in \mathcal{H} \times \pi_1(\Sigma)$ a *right-handed pair*. And h is said to be the *right-handed generator* of $\tau(\mathsf{H})$.

Lemma 7.3.2.

1. *For each halfspace $\mathfrak{h}$ of $\tilde{X}$, the right-handed generator is uniquely determined.*
2. *If h is the right-handed generator of a halfspace $\mathfrak{h}$, then h^{-1} is the right-handed generator of $\mathfrak{h}^*$.*
3. *If h is the right-handed generator of a halfspace $\mathfrak{h}$, then, for every $g \in \pi_1(\Sigma)$, the right-handed generator of $g\mathfrak{h}$ is ghg^{-1}.*

Proof. (1) Assume that h_1 and h_2 are right-handed generators of $\mathfrak{h}$. Since they preserves the boundary $\tilde{\gamma}$ of $\tau^{-1}(\mathfrak{h})$ and they are primitive, we have $h_1^n = h_2$ and $h_2^m = h_1$ for some integers n and m. So either $h_1 = h_2$ or $h_1 = h_2^{-1}$. For arbitrary point p in $\tilde{\gamma}$, the geodesics path from p to $h_1 p$ and the path from p to $h_2 p$ follow the orientation of $\tilde{\gamma}$. So we have $h_1 = h_2$.

(2) It is obvious by definition.

(3) Let $\tilde{\gamma}$ be the wall of $\tau^{-1}(\mathfrak{h})$. Since ghg^{-1} is primitive and $(ghg^{-1})g\tilde{\gamma} = g\tilde{\gamma}$, the conjugate of h by $\pi_1(\Sigma)$ is a generator of the stabilizer of $g\tilde{\gamma}$. Because $\pi_1(\Sigma)$ preserves the orientation of $\mathbb{H}^2$, we have $(g\mathfrak{h}, ghg^{-1})$ is a right-handed pair. □

Proposition 7.3.3. *If γ is a simple closed geodesic of $\mathcal{F}$ and h is the right-handed generator of a halfspace of $\tilde{X}_\gamma$, then the translation length of h with respect to the combinatorial metric on $\tilde{X}$ is $\sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)$. In particular, a vertex x is contained in the carrier of the hyperplane $\hat{\mathfrak{h}}$ if and only if $d(x,hx) = \operatorname{tr} h$.*

Proof. Let $\mathfrak{h}$ be the halfspace of $\tilde{X}$ such that h is the right-handed generator of $\mathfrak{h}$. And let x be a vertex on the carrier of $\hat{\mathfrak{h}}$. Because h preserves $\mathfrak{h}$, the hyperplane $\hat{\mathfrak{h}}$ does not separate x and hx. Since hx is also contained in the carrier of $\hat{\mathfrak{h}}$, every hyperplane separating x and hx crosses $\hat{\mathfrak{h}}$. So, by Lemma 7.2.6.(2), we have $\mathcal{I}(hx) - \mathcal{I}(x) = \{\mathfrak{k} \,|\, hx \in \mathfrak{k}, x \in \mathfrak{k}^*\} \subset \mathcal{GH}(h)$. Because each halfspace of $\mathcal{I}(hx) - \mathcal{I}(x)$ is contained in a different $\langle h\rangle$-orbit, we have $d(x,hx) \leq \operatorname{tr} h$ by Proposition 7.2.7. By the definition of translation length, $d(x,hx) = \operatorname{tr} h$.

Fix $\alpha \in \mathcal{F} - \{\gamma\}$. By Lemma 7.1.4, the intersection $K := \bigcap_{\mathfrak{k}\in\mathcal{I}(\pi_\alpha(x))} \tau_\alpha^{-1}(\mathfrak{k})$ is nonempty. Let $\tilde{\gamma}$ be the wall satisfying that $\tau(\tilde{\gamma}) = \hat{\mathfrak{h}}$. Because either $\mathfrak{h}$ or $\mathfrak{h}^*$ is a minimal halfspace of $\mathcal{I}(x)$, either $\tau^{-1}(\mathfrak{h})$ or $\tau^{-1}(\mathfrak{h}^*)$ intersects K but not properly. So $\tilde{\gamma}$ intersects K, and then there is a point p on $\tilde{\gamma}$ such that $\mathcal{I}(\pi_\alpha(x)) = \{\tau_\alpha(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}(\mathcal{L}_\alpha), p \in \mathsf{H}\}$. Let $[p, hp]$ denote the geodesic segment connecting p and hp. Because h is primitive, the interior of $[p,hp]$ embeds into Σ under the map $\xi : \mathbb{H}^2 \to \Sigma$. So the number of walls of $\mathcal{L}_\alpha$ which cross $[p,hp]$ is exactly $i(\gamma,\alpha)$. It implies that $d_\alpha(\pi_\alpha(x), h\pi_\alpha(x)) = i(\gamma,\alpha)$. Therefore, by Lemma 7.1.3, we have $d(x,hx) = \sum_{\alpha\in\mathcal{F}} d_\alpha(\pi_\alpha(x), \pi_\alpha(hx)) = \sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)$.

Suppose that x is not on the carrier of $\hat{\mathfrak{h}}$. For the closed-point projection $\pi_{\mathcal{N}(\hat{\mathfrak{h}})} : \tilde{X} \to \mathcal{N}(\hat{\mathfrak{h}})$, let y denote $\pi_{\mathcal{N}(\hat{\mathfrak{h}})}(x)$. Then every hyperplane separating y from hy also separates x from hx. So we have $d(x,hx) \geq d(y,hy) = \operatorname{tr} h$. Let $\mathfrak{k}$ be a halfspace such that $y \in \mathfrak{k}$ and $x \in \mathfrak{k}^*$. Since y is the projective image of x, the carrier of $\hat{\mathfrak{h}}$ is a subcomplex of $\mathfrak{k}$. Note that $h\hat{\mathfrak{k}}$ separates hx from hy and it is disjoint from $\hat{\mathfrak{k}}$. Since $\mathcal{N}(\hat{\mathfrak{h}}) \subset h\mathfrak{k}$, we have $\mathfrak{k} \supsetneq h\mathfrak{k}^*$. Since $hx \in h\mathfrak{k}^*$, it holds that $\hat{\mathfrak{k}}$ separates x from hx. But y and hy are contained in $\mathfrak{k}$. So $d(x,hx) > d(y,hy) = \operatorname{tr} h$. □

Proposition 7.3.4. *Let γ be a simple closed geodesic of $\mathcal{F}$. Let $\mathfrak{h}$ be a halfspace of $\tilde{X}$, and let h be its right-handed generator. If $\min_\gamma(h)$ is the subtree* *of* $\tilde{X}_\gamma$ *with the* 0*-skeleton* $\min_\gamma(h)^{(0)} = \{v \in \tilde{X}_\gamma^{(0)} \,|\, d_\gamma(v, hv) = \operatorname{tr}_\gamma h\}$*, then* $\min_\gamma(h)$ *is equal to* $\pi_\gamma(\mathcal{N}(\hat{\mathfrak{h}}))$.

Proof. Let α be a simple closed geodesic of $\mathcal{F}$ such that the axis of h on $\mathbb{H}^2$ is a lift of α. If $\alpha = \gamma$, then $\pi_\gamma(\mathcal{N}(\hat{\mathfrak{h}}))$ is the edge which is fixed by h. If α is disjoint from γ, then h fixes the unique vertex $\pi_\gamma(\hat{\mathfrak{h}}) = \pi_\gamma(\mathcal{N}(\hat{\mathfrak{h}}))$ on $\tilde{X}_\gamma$.

Assume that $i(\alpha, \gamma) > 0$. Without loss of generality, we assume that $\mathcal{F} = \{\alpha, \gamma\}$. (If $\{\alpha, \gamma\} \subsetneq \mathcal{F}$, then we consider $\pi_{\{\alpha,\gamma\}}(\mathfrak{h})$ and h.) Since h is hyperbolic on $\tilde{X}_\gamma$, the subtree $\min_\gamma(h)$ is a combinatorial geodesic, and the set $\mathcal{GH}(h, \gamma)$ is nonempty and totally ordered. Note that $\pi_\gamma^{-1}(\mathfrak{k}) \in \mathcal{GH}(h, \mathcal{F})$ for every halfspace $\mathfrak{k} \in \mathcal{GH}(h, \gamma)$. So

$$\{\pi_\gamma^{-1}(\mathfrak{k}) \,|\, \mathfrak{k} \in \mathcal{GH}(h, \gamma)\} \subseteq \mathcal{GH}(h, \mathcal{F}).$$

Conversely, for every halfspace $\mathfrak{l} \in \mathcal{GH}(h, \mathcal{F})$, the hyperplane $\hat{\mathfrak{l}}$ crosses $\hat{\mathfrak{h}}$. Then $\mathfrak{l}$ is not the preimage of a halfspace of $\tilde{X}_\alpha$, so $\pi_\gamma(\mathfrak{l})$ is a halfspace and $h\pi_\gamma(\mathfrak{l}) = \pi_\gamma(h\mathfrak{l}) \subsetneq \pi_\gamma(\mathfrak{l})$. It implies that $\pi_\gamma(\mathfrak{l}) \in \mathcal{GH}(h, \gamma)$, and

$$\{\pi_\gamma^{-1}(\mathfrak{k}) \,|\, \mathfrak{k} \in \mathcal{GH}(h, \gamma)\} \supseteq \mathcal{GH}(h, \mathcal{F}).$$

Hence, they are equal, and

$$\operatorname{tr}_\gamma h = \operatorname{tr}_{\mathcal{F}} h = i(\alpha, \gamma).$$

For every $x \in \mathcal{N}(\hat{\mathfrak{h}})$,

$$d_\gamma(\pi_\gamma(x), h\pi_\gamma(x)) \le d(x, hx) = \operatorname{tr} h = i(\alpha, \gamma) = \operatorname{tr}_\gamma h.$$

So $\pi_\gamma(x) \in \min_\gamma(h)$, and, therefore, $\pi_\gamma(\mathcal{N}(\hat{\mathfrak{h}})) \subseteq \min_\gamma(h)$. Let a vertex $v \in \min_\gamma(\mathrm{h})$ be given. If $\pi_\gamma^{-1}(v)$ is disjoint from $\mathcal{N}(\hat{\mathfrak{h}})$, there is a halfspace $\mathfrak{k}$ in the inward orientation of v such that $\pi_\gamma^{-1}(\mathfrak{k})$ is disjoint from $\mathcal{N}(\hat{\mathfrak{h}})$. So, for every vertex $y \in \pi_\gamma^{-1}(v)$, if z is the closest point of y on $\mathcal{N}(\hat{\mathfrak{h}})$, then $y \neq z$ and $d(y, hy) > d(z, hz) = i(\alpha, \gamma)$. Then it is a contradiction. It induces that there is a vertex y in $\mathcal{N}(\hat{\mathfrak{h}})$ such that $v = \pi_\gamma(y)$. Therefore, $\pi_\gamma(\mathcal{N}(\hat{\mathfrak{h}})) \supseteq \min_\gamma(h)$. $\square$

Chapter 8

Sliding quasi-isometries on dual cube complexes

Let Σ denote an oriented hyperbolic surface of finite area. And let $\mathcal{F}$ be a nonempty finite set of simple closed geodesics on Σ. Let $\tilde{X}$ be the dual cube complex obtained from the pair $(\mathbb{H}^2, \tilde{\mathcal{F}})$.

In Chapter 7, we observed that the fundamental group $\pi_1(\Sigma)$ of Σ acts faithfully and cocompactly on $\tilde{X}$ by isometries: see Proposition 7.1.6 and Lemma 7.1.9. So we identify $\pi_1(\Sigma)$ as a subgroup of the isometry group of $\tilde{X}$.

In this chapter, we will prove Theorem 2 that there exist $\mathcal{F}$ and a subgroup of the quasi-isometry group of $\tilde{X}$ (with respect to both the L^1-metric and the L^2-metric) such that it is isomorphic to the automorphism group of $\pi_1(\Sigma)$. Additionally, we prove in Appendix that the class of above quasi-isometries are lifts of Dehn twists in some way. (Theorem A.2.2)

8.1 The construction of sliding permutations

Let $\gamma \in \mathcal{F}$ and a vertex $v \in \tilde{X}_\gamma$ be given. For each $n \geq 1$, let

$$\mathcal{O}(v, n) := \{\mathfrak{h} \in \mathcal{H}_\gamma \,|\, \text{the distance between } v \text{ and } \pi_\gamma(\mathfrak{h}) \text{ is } n\}.$$

For example, if $\mathfrak{h}$ is a halfspace in $\mathcal{O}(v, 1)$, then $\pi_\gamma(\mathfrak{h})$ is a maximal element of the outward orientation $\mathcal{O}(v)$. Since $\tilde{X}_\gamma$ is a tree, all halfspaces in $\mathcal{O}(v, n)$ are pairwise disjoint.

For each $n \geq 1$, we define inductively a permutation $f_{v,n} : \tilde{X}^{(0)} \to \tilde{X}^{(0)}$ as follows:

$$f_{v,1}(x) = \begin{cases} x & \text{if } x \in \mathfrak{h}^* \text{ for all } \mathfrak{h} \in \mathcal{O}(v,1), \\ hx & \text{if } x \in \mathfrak{h} \text{ for some } \mathfrak{h} \in \mathcal{O}(v,1); \end{cases}$$

and, for each n,

$$f_{v,n}(x) = \begin{cases} f_{v,n-1}(x) & \text{if } x \in \mathfrak{h}^* \text{ for all } \mathfrak{h} \in \mathcal{O}(v,n), \\ f_{v,n-1}(hx) & \text{if } x \in \mathfrak{h} \text{ for some } \mathfrak{h} \in \mathcal{O}(v,n). \end{cases}$$

Remark 8.1.1. Every right-handed generator h of a halfspace $\mathfrak{h}$ moves $\mathfrak{h}$ to itself. It implies that each $f_{v,n}$ is indeed a permutation on $\tilde{X}^{(0)}$.

Remark 8.1.2. For every vertex x of $\tilde{X}$, if the distance between $\pi_\gamma(x)$ and v is N, then we can easily check that $f_{v,n}(x) = f_{v,N}(x)$ for all $n \geq N$.

Definition 8.1.3 (Sliding permutations)**.** Let a simple closed geodesic $\gamma \in \mathcal{F}$ and a vertex $v \in \tilde{X}_\gamma$ be given. Define a permutation $f_v : \tilde{X}^{(0)} \to \tilde{X}^{(0)}$ by

$$f_v(x) = f_{v,n}(x)$$

for sufficiently large $n = n(x)$. And we call f_v the *(right-handed) sliding permutation* centered at v.

Remark 8.1.4. Note that every tree is uniquely geodesic. So, for every pair of vertices u and v on $\tilde{X}_\gamma$, the set $\mathcal{O}(u) \cap \mathcal{I}(v)$ with the inclusion is a finite-length totally ordered set. Let us compute how the sliding permutation f_v sends a vertex x. If $\{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ is the set of halfspaces of $\tilde{X}_\gamma$ such that $\pi_\gamma(x) \in \mathfrak{h}_i$ and $v \in \mathfrak{h}_i^*$, and if each h_i is the right-handed generator of $\mathfrak{h}_i$, then we have

$$\begin{aligned} f_v(x) = f_{v,n}(x) &= f_{v,n-1}(h_n x) = f_{v,n-2}(h_{n-1} h_n x) \\ &= \cdots = f_{v,1}(h_2 \ldots h_n x) = h_1 \ldots h_n x \end{aligned}$$

by Definition 8.1.3.

Lemma 8.1.5. *Let a simple closed geodesic $\gamma \in \mathcal{F}$ and a vertex $v \in \tilde{X}_\gamma$ be given. If x and y are vertices of $\tilde{X}$ satisfying that $\pi_\gamma(x) = \pi_\gamma(y)$, then*

$$d(f_v(x), f_v(y)) = d(x, y).$$

Proof. Let $\mathcal{O}(v) \cap \mathcal{I}(\pi_\gamma(x)) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$. If h_i is the right-handed generator of $\mathfrak{h}_i$ for each $i = 1, \ldots, n$, then

$$d(f_v(x), f_v(y)) = d(h_1 \ldots h_n x, h_1 \ldots h_n y) = d(x, y)$$

by Remark 8.1.4. □

Remark 8.1.6. Let $\mathcal{F}'$ be an arbitrary nonempty subset of $\mathcal{F}$. If $\tilde{X}_{\mathcal{F}'}$ is the CAT(0) cube complex obtained from $\mathcal{F}'$, (i.e., the dual cube complex of the wallspace $(\mathbb{H}^2, \mathcal{L}_{\mathcal{F}'})$,) then the collapsing $\pi_{\mathcal{F}'} : \tilde{X} \to \tilde{X}_{\mathcal{F}'}$ is $\pi_1(\Sigma)$-equivariant and $\tilde{X}_{\mathcal{F}'}$ has its own sliding permutations.

Let γ be a simple closed geodesic in $\mathcal{F}'$ and choose a vertex $v \in \tilde{X}_\gamma$. If f_v and f'_v are sliding permutations centered at v defined on $\tilde{X}$ and $\tilde{X}_{\mathcal{F}'}$, respectively, then we have

$$\pi_{\mathcal{F}'} \circ f_v = f'_v \circ \pi_{\mathcal{F}'}.$$

In detail, for each vertex $x \in \tilde{X}$, we have $\pi_\gamma(x) = (\pi'_\gamma \circ \pi_{\mathcal{F}'})(x)$ where π'_γ is the collapsing from $\tilde{X}_{\mathcal{F}'}$ to $\tilde{X}_\gamma$. Then, by Remark 8.1.4, there is an element $g \in \pi_1(\Sigma)$, depending only on the set $\mathcal{O}(v) \cap \mathcal{I}(\pi_\gamma(x))$, such that $f_v(x) = gx$ and $f'_v(\pi_{\mathcal{F}'}(x)) = g\pi_{\mathcal{F}'}(x)$. So

$$\pi_{\mathcal{F}'} f_v(x) = \pi_{\mathcal{F}'}(gx) = g\pi_{\mathcal{F}'}(x) = f'_v \pi_{\mathcal{F}'}(x).$$

It means that the sliding permutation centered at v does not depend on the set $\mathcal{F}$ but only on γ. We will write all permutations centered at a vertex v by f_v.

8.2 Sliding permutations on trees

Tree with one simple closed geodesic

Given $\gamma \in \mathcal{F}$ and a vertex v on $\tilde{X}_\gamma$, we will first analyze how the sliding permutation f_v acts on $\tilde{X}_\gamma$.

Lemma 8.2.1. *Let $\gamma \in \mathcal{F}$ and a vertex $v \in \tilde{X}_\gamma$ be given.*

1. *Let u be a vertex of $\tilde{X}_\gamma$. If $\mathcal{O}(v) \cap \mathcal{I}(u) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$ for each i, then*

$$\mathcal{O}(v) \cap \mathcal{I}(f_v(u)) = \{\mathfrak{h}_1 \supsetneq h_1\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1 \ldots h_{n-1}\mathfrak{h}_n\}.$$

2. f_v on $\tilde{X}_\gamma$ is an isometry.

Proof. (1) Let $\vec{L} = (\vec{e}_1, \dots, \vec{e}_n)$ be the geodesic path from v to u. Then the terminal halfspace of each oriented edge $\vec{e}_i$ is $\mathfrak{h}_i$. For each i, let u_i be the terminal vertex of $\vec{e}_i$, and let u_0 denote v. Then, by Remark 8.1.4, we have $f_v(u_i) = h_1 \dots h_{i-1} u_i$. For each $i = 1, \dots, n$, since

$$\begin{aligned} d_\gamma(f_v(u_{i-1}), f_v(u_i)) &= d_\gamma(h_1 \dots h_{i-2} u_{i-1}, h_1 \dots h_{i-1} u_i) \\ &= d(u_{i-1}, h_{i-1} u_i) \\ &= d(h_{i-1} u_{i-1}, h_{i-1} u_i) = 1, \end{aligned}$$

there is an edge between $f_v(u_{i-1})$ and $f_v(u_i)$, which is exactly $h_1 \dots h_{i-1} \vec{e}_i$. So the sequence $(\vec{e}_1, h_1\vec{e}_2, \dots, h_1 \dots h_{n-1}\vec{e}_n)$ is a combinatorial path joining v to $f_v(u)$.

For each i, because $\mathfrak{h}_{i-1} \supsetneq h_{i-1}\mathfrak{h}_i$, we have

$$h_1 \dots h_{i-2}\mathfrak{h}_{i-1} \supsetneq h_1 \dots h_{i-2} h_{i-1} \mathfrak{h}_i.$$

It implies that the combinatorial path $(\vec{e}_1, \dots, h_1 \dots h_{n-1}\vec{e}_n)$ is geodesic. Hence, we conclude that $\mathcal{O}(v) \cap \mathcal{I}(f_v(u)) = \{\mathfrak{h}_1 \supsetneq h_1\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1 \dots h_{n-1}\mathfrak{h}_n\}$.

(2) Let u and w be distinct vertices. By (1), we have $d_\gamma(v, u) = d_\gamma(v, f_v(u))$ and $d_\gamma(v, w) = d_\gamma(v, f_v(w))$. Let μ be the median of u, v and w. We claim that $f_v(\mu)$ is the median of $f_v(u)$, v and $f_v(w)$.

For contradiction, assume that the median μ' of $f_v(u)$, v and $f_v(w)$ is not $f_v(\mu)$. Let L_u and L_w be the combinatorial geodesics joining v to u and w, respectively. By (1), the images of L_u and L_w under f_v are also combinatorial geodesics joining v to $f_v(u)$ and $f_v(w)$, respectively. Since $f_v(\mu)$ is contained in $f_v(L_u) \cap f_v(L_w)$, the vertex μ' is outside of the combinatorial geodesic connecting v to $f_v(\mu)$. Then there are two vertices u' and w' on the combinatorial geodesics joining μ to u and w such that $f_v(u') = \mu' = f_v(w')$. But it is a contradiction because f_v is bijective. So $f_v(\mu)$ is the median of $f_v(u)$, v and $f_v(w)$.

Note that $d_\gamma(f_v(u), f_v(\mu)) = d_\gamma(f_v(u), v) - d_\gamma(f_v(\mu), v) = d_\gamma(u, v) - d_\gamma(\mu, v) = d_\gamma(u, \mu)$ and, similarly, $d_\gamma(f_v(w), f_v(\mu)) = d_\gamma(w, \mu)$. Then

$$\begin{aligned} d_\gamma(f_v(u), f_v(w)) &= d_\gamma(f_v(u), f_v(\mu)) + d_\gamma(f_v(\mu), f_v(w)) \\ &= d_\gamma(u, \mu) + d_\gamma(\mu, w) = d_\gamma(u, w). \end{aligned}$$

Therefore, f_v is an isometry on $\tilde{X}_\gamma$. $\square$

The next lemma says that left-handed sliding permutations are the inverse of right-handed sliding permutations.

Lemma 8.2.2. *Let u and v be vertices of $\tilde{X}_\gamma$. If $\mathcal{O}(v) \cap \mathcal{I}(u) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$ for each i, then*

$$f_v^{-1}(u) = h_1^{-1} \dots h_n^{-1} u.$$

Furthermore,

$$\mathcal{O}(v) \cap \mathcal{I}(f_v^{-1}(u)) = \{\mathfrak{h}_1 \supsetneq h_1^{-1}\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1^{-1} \dots h_{n-1}^{-1}\mathfrak{h}_n\}.$$

Proof. Let $f_{v,l} : \tilde{X}_\gamma^{(0)} \to \tilde{X}_\gamma^{(0)}$ be defined by $f_{v,l}(u) = h_1^{-1} \dots h_n^{-1} u$ whenever $\mathcal{O}(v) \cap \mathcal{I}(u) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i is a right-handed generator of $\mathfrak{h}_i$. We claim that $f_{v,l} = f_v^{-1}$. Following the proof of Lemma 8.2.1.(1), it is obtained that $\mathcal{O}(v) \cap \mathcal{I}(f_v^{-1}(u)) = \{\mathfrak{h}_1 \supsetneq h_1^{-1}\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1^{-1} \dots h_{n-1}^{-1}\mathfrak{h}_n\}$. By Lemma 7.3.2.(3), the right-handed generator of $h_1^{-1} \dots h_{j-1}^{-1}\mathfrak{h}_j$ is

$$(h_1^{-1} \dots h_{j-1}^{-1}) h_j (h_1^{-1} \dots h_{j-1}^{-1})^{-1}.$$

So

$$\begin{aligned}
f_v f_{v,l}(u) &= f_v(f_{v,l}(u)) \\
&= h_1 (h_1^{-1} h_2 h_1) \dots (h_1^{-1} \dots h_{n-1}^{-1} h_n h_{n-1} \dots h_1) f_{v,l}(u) \\
&= h_n h_{n-1} \dots h_1 f_{v,l}(u) \\
&= (h_n h_{n-1} \dots h_1)(h_1^{-1} \dots h_n^{-1})(u) \\
&= u.
\end{aligned}$$

Similarly, because $\mathcal{O}(v) \cap \mathcal{I}(f_v(u)) = \{\mathfrak{h}_1 \supsetneq h_1\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1 \dots h_{m-1}\mathfrak{h}_m\}$, by Lemma 7.3.2.(3),

$$\begin{aligned}
f_{v,l} f_v(u) &= f_{v,l}(f_v(u)) \\
&= h_1^{-1} (h_1 h_2^{-1} h_1^{-1}) \dots (h_1 \dots h_{n-1} h_n^{-1} h_{n-1}^{-1} \dots h_1^{-1}) f_v(u) \\
&= h_n^{-1} h_{n-1}^{1} \dots h_1^{-1} f_v(u) \\
&= (h_n^{-1} h_{n-1}^{1} \dots h_1^{-1})(h_1 \dots h_n) u \\
&= u.
\end{aligned}$$

Hence, $f_{v,l}$ is the inverse of f_v. □

Remark 8.2.3. The following facts for powers of sliding permutations hold and we will acknowledge them without proof. For all vertices $v, u \in \tilde{X}_\gamma$, $x \in \tilde{X}$ and $n \in \mathbb{Z}$,

1. $f_v^n(x) = h_1^n \dots h_m^n x$ where $\mathcal{O}(u) \cap \mathcal{I}(\pi_\gamma(x)) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_m\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$.
2. $\mathcal{O}(v) \cap \mathcal{I}(f_v^n(u)) = \{\mathfrak{h} \supsetneq h_1^n \mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1^n \dots h_{m-1}^n \mathfrak{h}\}$.
3. f_v^n on $\tilde{X}_\gamma$ is an isometry.

Lemma 8.2.4. *For each vertex v of $\tilde{X}_\gamma$, the sliding permutation f_v centered at v fixes a vertex u of $\tilde{X}_\gamma$ if and only if $d_\gamma(u, v) \leq 1$.*

Proof. If $u = v$, then $f_v(u) = u$ by definition. If $d_\gamma(u, v) = 1$, then $\mathcal{O}(v) \cap \mathcal{I}(u)$ contains only one halfspace $\mathfrak{h}$. Because the right-handed generator h of $\mathfrak{h}$ fixes the edge joining u and v, we have $f_v(u) = hu = u$.

Assume that $d_\gamma(u, v) = n \geq 2$. If $\mathcal{O}(v) \cap \mathcal{I}(u) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i pairs right-handedly with $\mathfrak{h}_i$, then $f_v(u) \in h_1 \mathfrak{h}_2$ by Lemma 8.2.1.(1). Because $\mathfrak{h}_2$ is disjoint from $h_1 \mathfrak{h}_2$, we have $u \neq f_v(u)$. □

Since the actions of $\pi_1(\Sigma)$ on all dual cube complexes are faithful (Lemma 7.1.9,) we can consider $\pi_1(\Sigma)$ as a subgroup of the automorphism group of each dual cube complex.

Proposition 8.2.5. *Let Σ be an oriented hyperbolic surface of finite area. Let γ be a simple closed geodesic of Σ. If v and w are vertices of $\tilde{X}_\gamma$, then, for every $g \in \pi_1(\Sigma)$,*

1. $g f_v g^{-1} = f_{gv}$,
2. $f_w f_v^{-1} \in \pi_1(\Sigma)$,
3. $f_v g f_v^{-1} \in \pi_1(\Sigma)$ *and*
4. $f_w g f_v^{-1} \in \pi_1(\Sigma)$.

Furthermore, if $\mathcal{O}(w) \cap \mathcal{I}(v) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$ for each $i = 1, \dots, n$, then $f_w f_v^{-1} = h_1 \dots h_n$.

Proof. The statement (3) is the special case of (4). The statement (4) is immediately followed by (1) and (2). In detail, because $f_w g f_v^{-1} = (f_w f_{gv}^{-1})g$, we have $f_w g f_v^{-1} \in \pi_1(\Sigma)$ by (2). So we just need to prove (1) and (2).

(1) Choose a vertex u of $\tilde{X}_\gamma$. If $\mathcal{O}(gv) \cap \mathcal{I}(u) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$, then $\mathcal{O}(v) \cap \mathcal{I}(g^{-1}u) = \{g^{-1}\mathfrak{h}_1 \supsetneq \cdots \supsetneq g^{-1}\mathfrak{h}_n\}$. So

$$\begin{aligned} g f_v g^{-1}(u) &= g f_v(g^{-1}u) \\ &= g(g^{-1}h_1 g)\dots(g^{-1}h_n g)g^{-1}u \\ &= h_1 \dots h_n u \\ &= f_{gv}(u) \end{aligned}$$

by Lemma 7.3.2.(3). Hence, $g f_v g^{-1} = f_{gv}$.

(2) Let a vertex u of $\tilde{X}_\gamma$ be given. Let μ_0 be the median of u, v and w in $\tilde{X}_\gamma$. We put

$$\begin{aligned} \mathcal{O}(u) \cap \mathcal{I}(\mu_0) &= \{\mathfrak{h}_{u,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{u,n(u)}\}, \\ \mathcal{O}(v) \cap \mathcal{I}(\mu_0) &= \{\mathfrak{h}_{v,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{v,n(v)}\} \text{ and} \\ \mathcal{O}(w) \cap \mathcal{I}(\mu_0) &= \{\mathfrak{h}_{w,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)}\}. \end{aligned}$$

for some $n(u), n(v), n(w) \in \mathbb{N}$. For each i, let $h_{u,i}$, $h_{v,i}$ and $h_{w,i}$ be the right-handed generators of $\mathfrak{h}_{u,i}$, $\mathfrak{h}_{v,i}$ and $\mathfrak{h}_{w,i}$, respectively. Since the combinatorial geodesic joining v to u contains μ_0,

$$\begin{aligned} \mathcal{O}(v) \cap \mathcal{I}(u) &= (\mathcal{O}(v) \cap \mathcal{I}(\mu_0)) \cup (\mathcal{O}(\mu_0) \cap \mathcal{I}(u)) \\ &= \{\mathfrak{h}_{v,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{v,n(v)} \supsetneq \mathfrak{h}^*_{u,n(u)} \supsetneq \cdots \supsetneq \mathfrak{h}^*_{u,1}\}. \end{aligned}$$

So $f_v^{-1}(x) = h_{v,1}^{-1} \dots h_{v,n(v)}^{-1} h_{u,n(u)} \dots h_{u,1} x$. On the other hand, we have

$$\begin{aligned} \mathcal{O}(w) \cap \mathcal{I}(v) &= (\mathcal{O}(w) \cap \mathcal{I}(\mu_0)) \cup (\mathcal{O}(\mu_0) \cap \mathcal{I}(v)) \\ &= \{\mathfrak{h}_{w,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)} \supsetneq \mathfrak{h}^*_{v,n(v)} \supsetneq \cdots \supsetneq \mathfrak{h}^*_{v,1}\}. \end{aligned}$$

We claim that $f_w f_v^{-1}(u) = h_{w,1} \dots h_{w,n(w)} h_{v,n(v)}^{-1} \dots h_{v,1}^{-1}(u)$.

Before we start proving this claim, we notice that some sets defined in the proof can be empty. Then each empty set will correspond to the empty word, i.e., the identity. For example, if $v = \mu_0$, then $\mathcal{O}(v) \cap \mathcal{I}(\mu_0)$ is empty. In this case, our assertion is that $f_w f_v^{-1}(u) = h_{w,1} \dots h_{w,m_w}(u)$.

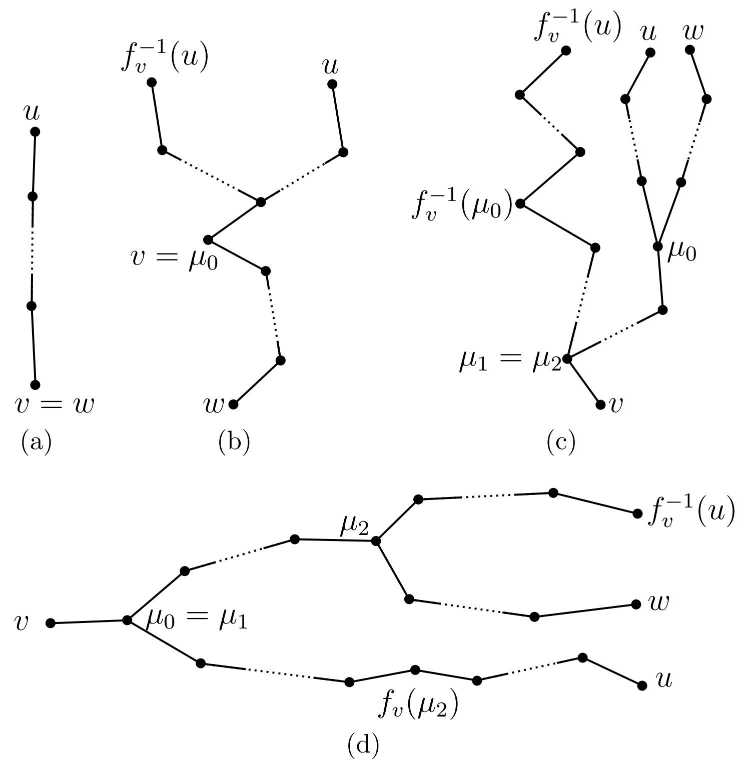

Figure 8.1: Samples in the proof of Proposition 8.2.5.(2): Figure 8.1a is an example of the case that $v = w$; Figure 8.1b is an example of the case that $v \neq w$ and $v = \mu_0$; Figure 8.1c is an example of the case that $v \neq w$, $v \neq \mu_0$ and $\mu_1 = \mu_2$; Figure 8.1d is an example of the case that $v \neq w$, $v \neq \mu_0$ and $\mu_1 \neq \mu_2$.

We divide the proof of the claim into four cases. Each tree in Figure 8.1 is an example of each case.

(a) If $v = w$, then $\mu_0 = v = w$ so that both $\mathcal{O}(v) \cap \mathcal{I}(\mu_0)$ and $\mathcal{O}(w) \cap \mathcal{I}(\mu_0)$ are empty. Since $f_w f_v^{-1}$ is the identity, the claim is true in this case.

(b) Assume that $v \neq w$ and $v = \mu_0$. Then $\mathcal{O}(v) \cap \mathcal{I}(\mu_0)$ is empty and

$$\begin{aligned}
&\mathcal{O}(v) \cap \mathcal{I}(f_v^{-1}(u)) \\
&\quad = \mathcal{O}(\mu_0) \cap \mathcal{I}(f_v^{-1}(u)) \\
&\quad = \{\mathfrak{h}^*_{u,n(u)} \supsetneq h_{u,n(u)}\mathfrak{h}^*_{u,n(u)-1} \supsetneq \cdots \supsetneq h_{u,n(u)} \dots h_{u,2}\mathfrak{h}^*_{u,1}\}
\end{aligned}$$

by Lemma 8.2.1.(1). The edge containing the midpoint $\hat{\mathfrak{h}}_{u,n(u)}$ lies on the combinatorial geodesic joining μ_0 and $f_v^{-1}(u)$. (See Figure 8.1b.) So the combinatorial geodesic joining w and $f_v^{-1}(u)$ contains μ_0. Since $\mathcal{O}(w) \cap \mathcal{I}(f_v^{-1}(u)) = (\mathcal{O}(w) \cap \mathcal{I}(\mu_0)) \cup (\mathcal{O}(\mu_0) \cap \mathcal{I}(f_v^{-1}(u)))$, we have

$$\begin{aligned}\mathcal{O}(w) \cap \mathcal{I}(f_v^{-1}(u)) = \{&\mathfrak{h}_{w,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)} \\ &\supsetneq \mathfrak{h}^*_{u,n(u)} \supsetneq h_{u,n(u)}\mathfrak{h}^*_{u,n(u)-1} \supsetneq \cdots \supsetneq h_{u,n(u)} \dots h_{u,2}\mathfrak{h}^*_{u,1}\}.\end{aligned}$$

Then, using Lemma 7.3.2.(3) and Remark 8.1.4, we have

$$\begin{aligned} f_w f_v^{-1}(u) &= h_{w,1} \dots h_{w,n(w)} \\ &\quad\cdot h^{-1}_{u,n(u)}(h_{u,n(u)}h^{-1}_{u,n(u)-1}h^{-1}_{u,n(u)}) \dots \\ &\quad\cdot (h_{u,n(u)} \dots h_{u,2}h^{-1}_{u,1}(h_{u,n(u)} \dots h_{u,2})^{-1}) f_v^{-1}(u) \\ &= h_{w,1} \dots h_{w,n(w)} u. \end{aligned}$$

(c) Assume that $v \neq w$ and $v \neq \mu_0$. Let μ_1 be the median of v, μ_0 and $f_v^{-1}(\mu_0)$. Because μ_1 is fixed by f_v^{-1}, it holds that μ_1 lies in the star of v by Lemma 8.2.4. Then $\mathfrak{h}_{v,1}$ exists, by the assumption, and $\mu_0, f_v^{-1}(\mu_0) \in \mathfrak{h}_{v,1}$. So the combinatorial geodesic joining μ_0 and $f_v^{-1}(\mu_0)$ does not pass through v. See Figure 8.1c. It implies that $\mu_1 \neq v$ and $\mathcal{O}(v) \cap \mathcal{I}(\mu_1) = \{\mathfrak{h}_{v,1}\}$.

Let μ_2 be the median of w, $f_v^{-1}(u)$ and μ_1. If $\mu_2 = \mu_1$, then μ_1 is on the combinatorial geodesic connecting w and $f_v^{-1}(u)$. So

$$\begin{aligned}\mathcal{O}(w) \cap \mathcal{I}(f_v^{-1}(u)) &= (\mathcal{O}(w) \cap \mathcal{I}(\mu_1)) \cup (\mathcal{O}(\mu_1) \cap \mathcal{I}(f_v^{-1}(u))) \\ &= (\mathcal{O}(w) \cap \mathcal{I}(\mu_0)) \cup (\mathcal{O}(\mu_0) \cap \mathcal{I}(\mu_1)) \\ &\quad \cup (\mathcal{O}(\mu_1) \cap \mathcal{I}(f_v^{-1}(\mu_0))) \cup (\mathcal{O}(f_v^{-1}(\mu_0)) \cap \mathcal{I}(f_v^{-1}(u)))\end{aligned}$$

Then

$$\begin{aligned}\mathcal{O}(w) \cap \mathcal{I}(f_v^{-1}(u)) = \{&\mathfrak{h}_{w,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)} \supsetneq \mathfrak{h}^*_{v,n(v)} \supsetneq \cdots \supsetneq \mathfrak{h}^*_{v,2} \\ &\supsetneq h^{-1}_{v,1}\mathfrak{h}_{v,2} \supsetneq \cdots \supsetneq h^{-1}_{v,1} \dots h^{-1}_{v,n(v)-1}\mathfrak{h}_{v,n(v)} \\ &\supsetneq h^{-1}_{v,1} \dots h^{-1}_{v,n(v)}\mathfrak{h}^*_{u,n(u)} \supsetneq \cdots \supsetneq h^{-1}_{v,1} \dots h^{-1}_{v,n(v)} h_{u,n(u)} \dots h_{u,2}\mathfrak{h}^*_{u,1}\}.\end{aligned}$$

Then, by calculation with Lemma 7.3.2.(3) and Remark 8.1.4, it is obtained that

$$f_w f_v^{-1}(u) = h_{w,1} \dots h_{w,n(w)} h^{-1}_{v,n(v)} \dots h^{-1}_{v,1} u.$$

(d) Finally, the case that $v \neq w$, $v \neq \mu_0$ and $\mu_1 \neq \mu_2$ remains only. We claim that $\mu_0 = \mu_1$ in this case. If not, μ_1 lies between v and μ_0. Because both μ_0 and μ_2 lie on the combinatorial geodesic joining μ_1 and w, there is a halfspace $\mathfrak{h}$ in $\mathcal{O}(\mu_1)\cap\mathcal{I}(\mu_2)$ such that $\mu_0 \in \mathfrak{h}$. So $\mathfrak{h}$ contains the combinatorial geodesic joining u and w as a subset. Then $f_v^{-1}(u)$ is contained in $f_v^{-1}\mathfrak{h}$. On the other hand, $\mathfrak{h}$ contains the combinatorial geodesic joining w and $f_v^{-1}(u)$ because $\mathfrak{h} \in \mathcal{O}(\mu_1)\cap\mathcal{I}(\mu_2)$. So $f_v^{-1}(u) \in \mathfrak{h}\cap f_v^{-1}\mathfrak{h}$. But it gives a contradiction because $\mathfrak{h}$ is disjoint to $f_v^{-1}\mathfrak{h}$. (Precisely, Since the edge of the midpoint $\hat{\mathfrak{h}}$ is not on the star of v, we have $\mathfrak{h}$ is not fixed by f_v^{-1}. And, because both $\mathfrak{h}$ and $f_v^{-1}\mathfrak{h}$ belong to the outward orientation of v, they are disjoint from each other.) So our claim is proved.

By the above claim, it holds that $\mathcal{O}(v)\cap\mathcal{I}(\mu_0) = \{\mathfrak{h}_{v,1}\}$. Because $\mathcal{O}(\mu_2)\cap\mathcal{I}(\mu_0)$ is a subset of $\mathcal{O}(f_v^{-1}(u))\cap\mathcal{O}(w)\cap\mathcal{I}(\mu_0)$, there is an integer $1 \leq N \leq \min\{n(u), n(w)\}$ such that

$$\mathcal{O}(\mu_2)\cap\mathcal{I}(\mu_0) = \{\mathfrak{h}_{w,n(w)-N+1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)}\}.$$

Since $\mathcal{O}(\mu_2)\cap\mathcal{I}(\mu_0) \subseteq \mathcal{O}(f_v^{-1}(u))\cap\mathcal{I}(\mu_0)$ and $f_v^{-1}(\mu_0) = \mu_0$, we have $\mathcal{O}(f_v(\mu_2))\cap\mathcal{I}(\mu_0) \subseteq \mathcal{O}(u)\cap\mathcal{I}(\mu_0)$ and $\mathcal{O}(u)\cap\mathcal{I}(\mu_0) = (\mathcal{O}(u)\cap\mathcal{I}(f_v(\mu_2)))\cup(\mathcal{O}(f_v(\mu_2))\cap\mathcal{I}(\mu_0))$. Then

$$\begin{aligned}\mathcal{O}(\mu_0)\cap\mathcal{I}(f_v(\mu_2)) = \{&h_{v,1}\mathfrak{h}^*_{w,n(w)} \supsetneq h_{v,1}h^{-1}_{w,n(w)}\mathfrak{h}^*_{w,n(w)-1} \supsetneq \cdots \\ &\supsetneq h_{v,1}h^{-1}_{w,n(w)}\dots h^{-1}_{w,n(w)-N+2}\mathfrak{h}^*_{w,n(w)-N+1}\}\end{aligned}$$

and $\mathcal{O}(u)\cap\mathcal{I}(f_v(\mu_2)) = \{\mathfrak{h}_{u,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{u,n(u)-N}\}$. So we have $h_{u,n(u)} = h_{v,1}h_{w,n(w)}h^{-1}_{v,1}$ and

$$\begin{aligned}h_{u,n(u)-i} = {}&(h_{v,1}h^{-1}_{w,n(w)}\dots h^{-1}_{w,n(w)-i+1})h_{w,n(w)-i}\\ &\cdot(h_{v,1}h^{-1}_{w,n(w)}\dots h^{-1}_{w,n(w)-i+1})^{-1}\end{aligned}$$

for each $1 \leq i \leq N-1$. From these equalities, we obtain that

$$f_v^{-1}(u) = h_{w,n(w)-N+1}\dots h_{w,n(w)}h^{-1}_{v,1}h_{u,n(u)-N}\dots h_{u,1}u$$

and

$$\begin{aligned}f_v^{-1}\mathfrak{h}_{u,n(u)-N-i} = {}&h_{w,n(w)-N+1}\dots h_{w,n(w)}h^{-1}_{v,1}\\ &\cdot h_{u,n(u)-N}\dots h_{u,n(u)-N-i+1}\mathfrak{h}_{u,n(u)-N-i}\end{aligned}$$

for each $i \in \{1, \ldots, n(u) - N + 1\}$. It implies that

$$\begin{aligned}
&\mathcal{O}(w) \cap \mathcal{I}(f_v^{-1}(u)) \\
&\quad = (\mathcal{O}(w) \cap \mathcal{I}(\mu_2)) \cup (\mathcal{O}(\mu_2) \cap \mathcal{I}(f_v^{-1}(u))) \\
&\quad = \{\mathfrak{h}_{w,1} \supsetneq \cdots \supsetneq \mathfrak{h}_{w,n(w)-N} \\
&\qquad \supsetneq h_{w,n(w)-N+1} \ldots h_{w,n(w)} h_{v,1}^{-1} \mathfrak{h}^*_{u,n(u)-N} \supsetneq \cdots \\
&\qquad \supsetneq h_{w,n(w)-N+1} \ldots h_{w,n(w)} h_{v,1}^{-1} h_{u,n(u)-N} \ldots h_{u,2} \mathfrak{h}^*_{u,1}\}.
\end{aligned}$$

By calculation, we obtain that

$$f_w f_v^{-1}(u) = h_{w,1} \ldots h_{w,n(w)} h_{v,1}^{-1} u.$$

So our proof is done. □

Proposition 8.2.5.(3) implies that the conjugation by a sliding permutation on $\pi_1(\Sigma)$ is an automorphism of $\pi_1(\Sigma)$. If a homeomorphism $F : \mathbb{H}^2 \to \mathbb{H}^2$ is a lift of a self-homeomorphism of Σ, then it induces an automorphism of $\pi_1(\Sigma)$ by $g \mapsto FgF^{-1}$. The next theorem and corollary means that sliding permutations are not distinguished from lifts of Dehn twists under the automorphism group of $\pi_1(\Sigma)$.

Theorem 8.2.6 (Theorem A.2.2)**.** *For a simple closed geodesic γ and every vertex v of $\tilde{X}_\gamma$, there is a lift $\tilde{T}_\gamma$ of a Dehn twist along γ such that $\Phi_{\gamma,\epsilon} \circ \tilde{T}_\gamma = f_v \circ \Phi_{\gamma,\epsilon}$ where $\Phi_{\gamma,\epsilon}$ is the $\pi_1(\Sigma)$-equivariant surjective continuous map defined in Proposition 3.1.3.*

Corollary 8.2.7 (Corollary A.2.3)**.** *If γ is a simple closed geodesic and v is a vertex of $\tilde{X}_\gamma$, there is a lift $\tilde{T}_\gamma$ of a Dehn twist along γ such that the automorphism $g \mapsto f_v g f_v^{-1}$ is equal to the automorphism $g \mapsto \tilde{T}_\gamma g \tilde{T}_\gamma^{-1}$.*

Lemma 8.2.8. *If u and v are vertices of $\tilde{X}_\gamma$, then there is $g_0 \in \pi_1(\Sigma)$ such that $f_v = f_u g_0$.*

Proof. If γ is a non-separating simple closed geodesic, then $\pi_1(\Sigma)$ acts on $\tilde{X}_\gamma$ transitively by Proposition 7.1.11 so that $v = gu$ for some $g \in \pi_1(\Sigma)$. So $f_v = f_{gu} = g f_u g^{-1} = f_u (f_u^{-1} g f_u) g^{-1}$. By Proposition 8.2.5.(3), we have $f_u^{-1} g f_u \in \pi_1(\Sigma)$. If g_0 denotes $(f_u^{-1} g f_u) g^{-1}$, then we have $f_v = f_u g_0$.

Assume that γ is separating. If $v = g'u$ for some $g' \in \pi_1(\Sigma)$, then it can be proved as above that $f_v = f_u g''$ for some g''. So suppose that there is no element of $\pi_1(\Sigma)$ which sends u to v. Since the orbit of an edge covers $\tilde{X}_\gamma$ by Remark 7.1.10, there is $g \in \pi_1(\Sigma)$ such that gu and v are joined by an edge. Let $\mathfrak{h}$ be the halfspace of $\tilde{X}_\gamma$ such that $gu \in \mathfrak{h}$ and $v \in \mathfrak{h}^*$. If h is the right-handed generator of $\mathfrak{h}$, we claim that $f_v = hf_{gu}$.

Choose a vertex w on $\tilde{X}_\gamma$. If $w = v$, then $\mathcal{O}(gu) \cap \mathcal{I}(w) = \{\mathfrak{h}^*\}$ and

$$hf_{gu}(w) = h(h^{-1}w) = w = f_v(w).$$

If $w = gu$, then $f_v(w) = hw = hf_{gu}(w)$. Otherwise, let $\mathcal{O}(v) \cap \mathcal{I}(w) =: \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}\}_n\}$, and let h_i be the right-handed generator of $\mathfrak{h}_i$. If $\mathfrak{h}_1 \neq \mathfrak{h}$, then $\mathcal{O}(gu) \cap \mathcal{I}(w) = \{\mathfrak{h}^* \supsetneq \mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$. So $f_v(w) = h_1 \ldots h_n w = h(h^{-1}h_1 \ldots h_n w) = hf_{gu}(w)$. Whenever $\mathfrak{h}_1 = \mathfrak{h}$, we have $\mathcal{O}(gu) \cap \mathcal{I}(w) = \{\mathfrak{h}_2 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$. It implies that $f_v(w) = hh_2 \ldots h_n w = hf_{gu}(w)$. So we proved the claim.

By Proposition 8.2.5.(1) and (3), we have

$$f_v = hf_{gu} = hgf_u g^{-1} = f_u(f_u^{-1}hf_u)(f_u^{-1}gf_u).$$

Therefore, if $g_0 = (f_u^{-1}hf_u)(f_u^{-1}gf_u)$, we have $f_v = f_u g_0$. □

Trees obtained from disjoint simple closed geodesics

Until now, we studied sliding permutations on the tree obtained from a single simple closed geodesics. Note that all simple closed geodesics in $\mathcal{F}$ are pairwise disjoint if and only if the dual cube complex $\tilde{X}$ of $\mathcal{F}$ is a tree. In this subsection, we suppose that $\tilde{X}$ is a tree.

Proposition 8.2.9. *For a set $\mathcal{F}$ of finitely many pairwise disjoint simple closed geodesics, let $\tilde{X}$ be the dual cube complex of $\mathcal{F}$. Then, for every simple closed geodesic $\gamma \in \mathcal{F}$, a vertex $v \in \tilde{X}_\gamma$ and an integer $n \in \mathbb{Z}$, the n-th power of sliding permutation f_v^n defined on $\tilde{X}$ is an isometry.*

The proof of Proposition 8.2.9 is similar to one of Lemma 8.2.1.(2). Note that we use notations d and d_γ as the combinatorial metrics on $\tilde{X}$ and $\tilde{X}_\gamma$, respectively.

And $\pi_\gamma : \tilde{X} \to \tilde{X}_\gamma$ is the collapsing to $\tilde{X}_\gamma$. (See Proposition 6.1.2 and Definition 6.1.3.)

Sketch of proof of Proposition 8.2.9. It is convenient to split the proof into following three assertions.

(i) For all vertices $x \in \tilde{X}$ and $z_0 \in \pi_\gamma^{-1}(v)$, we have $d(f_v^n(x), z_0) = d(x, z_0)$.

(ii) For all vertices $x, y \in \tilde{X}$ and $z_0 \in \pi_\gamma^{-1}(v)$, if μ is the median of x, y and z_0, then $f_v^n(\mu)$ is the median of $f_v^n(x)$, $f_v^n(y)$ and z_0.

(iii) For all vertices $x, y \in \tilde{X}$, we have $d(f_v^n(x), f_v^n(y)) = d(x, y)$.

(i) If $\pi_\gamma(x) = v$, then $f_v^n(x) = x$ and $d(f_v^n(x), z_0) = d(x, z_0)$. Suppose that $\pi_\gamma(x) \neq v$. Then there are oriented edges $\vec{e}_1, \dots, \vec{e}_m$ of $\tilde{X}$ such that the sequence $\pi_\gamma(\vec{e}_1), \dots, \pi_\gamma(\vec{e}_m)$ is the geodesic path from v to $\pi_\gamma(x)$. For each $i = 1, \dots, m$, let h_i be the right-handed generator of the terminal halfspace $\mathfrak{h}_i$ of $\vec{e}_i$. Note that $\pi_\gamma(\mathfrak{h}_i)$ is a halfspace of $\tilde{X}_\gamma$. By Remark 8.2.3, we have $\mathcal{O}(v) \cap \mathcal{I}(\pi_\gamma(x)) = \{\pi_\gamma(\mathfrak{h}_1) \supsetneq h_1\pi_\gamma(\mathfrak{h}_2) \supsetneq \cdots \supsetneq h_1 \dots h_{m-1}\pi_\gamma(\mathfrak{h}_m)\}$.

The distance between z_0 and x can be written as follow.

$$\begin{aligned} d(z_0, x) &= d(z_0, \vec{e}_1) + l(\vec{e}_1) + d(\vec{e}_1, \vec{e}_2) + \cdots + l(\vec{e}_m) + d(\vec{e}_m, x) \\ &= n + d(z_0, \vec{e}_1) + d(\vec{e}_m, x) + \sum_{i=1}^{m-1} d(\vec{e}_i, \vec{e}_{i+1}) \end{aligned}$$

where $l(\vec{e}_i)$ is the length of $\vec{e}_i$. Then the following facts hold.

- For each $i = 1, \dots, m$, since $h_1 \dots h_{i-1}^n \mathfrak{h}_i^n \in \mathcal{O}(z_0) \cap \mathcal{I}(f_v^n(x))$, the oriented edge $h_1^n \dots h_{i-1}^n \vec{e}_i$ belongs to the geodesic path from z_0 to $f_v^n(x)$.
- For each i, we have $d(h_1^n \dots h_{i-1}^n \vec{e}_i, h_1^n \dots h_i^n \vec{e}_{i+1}) = d(\vec{e}_i, \vec{e}_{i+1})$ because h_i fixes e_i.
- $d(h_1^n \dots h_{m-1}^n \vec{e}_m, h_1^n \dots h_m^n x) = d(\vec{e}_m, x)$.

So, when we calculate the distance between z_0 and $f_v^n(x)$,

$$\begin{aligned} d(z_0, f_v^n(x)) &= d(z_0, \vec{e}_1) + l(\vec{e}_1) + d(\vec{e}_1, h_1^n \vec{e}_2) + \cdots \\ &\qquad + l(h_1^n \dots h_{m-1}^n \vec{e}_m) + d(h_1^n \dots h_{m-1}^n \vec{e}_m, h_1^n \dots h_m^n x) \\ &= d(z_0, x). \end{aligned}$$

(ii) Let L_x and L_y be the combinatorial geodesics which connects z_0 with x and y, respectively. Then $L_x \cap L_y$ is the combinatorial geodesic joining z_0 and μ. By (i), it holds that $f_v^n(L_x)$ and $f_v^n(L_y)$ are also the combinatorial geodesics. Then $f_v^n(\mu) \in f_v^n(L_x) \cap f_v^n(L_y)$. If $f_v^n(\mu)$ is not the median of $f_v^n(x)$,

$f_v^n(y)$ and z_0, then $f_v^n(\mu)$ is not an endpoint of $f_v^n(L_x) \cap f_v^n(L_y)$. It implies that two distinct edges which share the vertex μ has the same image via f_v^n. It is a contradiction because f_v^n is bijective. Hence, $f_v^n(\mu)$ is the median of $f_v^n(x)$, $f_v^n(y)$ and z_0.

(iii) If μ is the median of x, y and z_0, then

$$\begin{aligned} d(f_v^n(x), f_v^n(y)) &= d(f_v^n(x), f_v^n(\mu)) + d(f_v^n(\mu), f_v^n(y)) \\ &= (d(f_v^n(x), z_0) - d(f_v^n(\mu), z_0)) + (d(z_0, f_v^n(y)) - d(z_0, f_v^n(\mu))) \\ &= (d(x, z_0) - d(\mu, z_0)) + (d(z_0, y) - d(z_0, \mu)) \\ &= d(x, y) \end{aligned}$$

by (i) and (ii). □

Corollary 8.2.10. *If α and β are disjoint simple closed geodesics, then, for every vertex $v \in \tilde{X}_\alpha$ and an integer $n \in \mathbb{Z}$, there is an isometry f on $\tilde{X}_\beta$ such that $f(\pi_\beta(x)) = \pi_\beta(f_v^n(x))$ for each vertex x on the dual cube complex of $\{\alpha, \beta\}$.*

Proof. Let $\tilde{X}$ be the dual cube complex of $\{\alpha, \beta\}$. Because f_v^n is an isometry on $\tilde{X}$, it permutes halfspaces of $\tilde{X}$. Note that every halfspace of $\tilde{X}$ is the inverse image of a halfspace of either $\tilde{X}_\alpha$ or $\tilde{X}_\beta$. Because f_v^n permutes all halfspaces induced from $\tilde{X}_\alpha$, it also permutes the others, all halfspaces induced from $\tilde{X}_\beta$. So it is an isometry of $\tilde{X}_\beta$. □

Proposition 8.2.11. *Let α and β be disjoint simple closed geodesics. If u and v are vertices of $\tilde{X}_\alpha$ and $\tilde{X}_\beta$, respectively, and if n and m are integers, then the following are satisfied.*

1. *If $\mathfrak{h}$ is a halfspace of $\tilde{X}_\beta$ and h is the right-handed generator of $\mathfrak{h}$, then the right-handed generator of $f_u^n\mathfrak{h}$ is $f_u^n h f_u^{-n}$.*

2. *The conjugate of f_v^m by f_u^n, which is $f_u^n f_v^m f_u^{-n}$, is the m-th power of the sliding permutation centered at $f_u^n(v)$, i.e., $f_u^n f_v^m f_u^{-n} = f_{f_u^n(v)}^m$.*

3. *The commutator of f_u^n and f_v^m, denoted by $[f_u^n, f_v^m] := f_u^n f_v^m f_u^{-n} f_v^{-m}$, is the element of $\pi_1(\Sigma)$.*

Furthermore, if $\tilde{X}$ is the dual cube complex of $\{\alpha, \beta\}$ and z_0 is a vertex such that $u = \pi_\alpha(z_0)$ and $v = \pi_\mu(z_0)$, then we have $[f_u^n, f_v^m] = 1$.

Proof. (1) Let $\tilde{X}$ be the dual cube complex of $\{\alpha, \beta\}$, and let $\vec{e}$ be the oriented edge of $\tilde{X}$ such that $\pi_\beta(\vec{e})$ is the oriented edge whose terminal halfspace is $\mathfrak{h}$. Then $\pi_\alpha(\vec{e})$ is a vertex. Let $\{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_l\}$ be the set $\mathcal{O}(u) \cap \mathcal{I}(\pi_\alpha(\vec{e}))$. By Remark 8.2.3, we have $f_u^n \vec{e} = h_1^n \dots h_l^n \vec{e}$. Because $f_u^n \pi_\beta^{-1}(\mathfrak{h})$ is the terminal halfspace of $f_u^n \vec{e}$, we have $f_u^n \mathfrak{h} = h_1^n \dots h_l^n \mathfrak{h}$.

Second, we claim that h fixes a unique vertex on $\tilde{X}_\alpha$. Note that h fixes $\pi_\alpha(\vec{e})$. If another vertex u' on $\tilde{X}_\alpha$ is fixed by h, then h must fix the geodesic joining $\pi_\alpha(\vec{e})$ and u' pointwise. So h fixes some edge of $\tilde{X}_\alpha$, which induces that there is a lift of α which is the axis of h. It is a contradiction. That is, the uniqueness holds.

Because h must fix the median of u, hu and $\pi_\alpha(\vec{e})$, the vertex $\pi_\alpha(\vec{e})$ lies on the combinatorial geodesic joining u and hu by uniqueness. Note that $f_u^n \mathfrak{h}$ is a halfspace of $\tilde{X}_\beta$ because of Corollary 8.2.10. By Proposition 8.2.5.(3), we have $f_u^n h f_u^{-n} \in \pi_1(\Sigma)$. Then

$$\begin{aligned} \mathcal{O}(u) \cap I(hu) &= \big(\mathcal{O}(u) \cap \mathcal{I}(\pi_\alpha(\vec{e}))\big) \cup \big(\mathcal{O}(h\pi_\alpha(\vec{e})) \cap \mathcal{I}(hu)\big) \\ &= \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_l \supsetneq h\mathfrak{h}_l^* \supsetneq \cdots \supsetneq h\mathfrak{h}_1^*\}. \end{aligned}$$

If h_i is the right-handed generator of $\mathfrak{h}_i$ for each i, then

$$\begin{aligned} f_u^n h f_u^{-n} &= f_u^n f_{hu}^{-n} h \\ &= (h_1^n \dots h_l^n)(h h_l^{-n} h^{-1}) \dots (h h_1^{-n} h^{-1}) h \\ &= (h_1^n \dots h_l^n) h (h_1^n \dots h_l^n)^{-1} \end{aligned}$$

by Proposition 8.2.5.(1) and Remark 8.2.3. Therefore, $f_u^n h f_u^{-n}$ is the right-handed generator of $f_u^n \mathfrak{h}$ by Lemma 7.3.2.

(2) Choose a vertex w of $\tilde{X}_\beta$. If $\mathcal{O}(f_u^n(v)) \cap \mathcal{I}(w) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_l\}$ and h_i is the right-handed generator of $\mathfrak{h}_i$ for each $i = 1, \dots, l$, then we have

$$f_{f_u^m(v)}^m(w) = h_1^m \dots h_l^m w.$$

On the other hand, because f_u^n is an isometry on $\tilde{X}_\beta$,

$$\mathcal{O}(v) \cap \mathcal{I}(f_u^{-1}(w)) = \{f_u^{-n}\mathfrak{h}_1 \supsetneq \cdots \supsetneq f_u^{-n}\mathfrak{h}_l\}.$$

By (1),

$$f_u^n f_v^m f_u^{-n}(w) = f_u^n (f_u^{-n} h_1^m f_u^n) \cdots (f_u^{-n} h_l^m f_u^n) w$$

$$= h_1^m \dots h_l^m w = f^m_{f_u^n(v)}(w).$$

Because w is an arbitrary vertex of $\tilde{X}_\beta$, we have $f_u^n f_v^m f_u^{-n} = f^m_{f_u^n}$.

(3) By (2), we have $[f_u^n, f_v^m] = f_u^n f_v^m f_u^{-n} f_v^{-m} = f^m_{f_u^n(v)} f_v^{-m}$. So it is an element of $\pi_1(\Sigma)$ by Proposition 8.2.5.(2). If $u = \pi_\alpha(z_0)$ and $v = \pi_\beta(z_0)$ for some $z_0 \in \tilde{X}$, then $f_u^n(v) = f_u^n(\pi_\beta(z_0)) = \pi_\beta(f_u^n(z_0)) = \pi_\beta(z_0) = v$. So $[f_u^n, f_v^m] = f^m_{f_u^n(v)} f_v^{-m} = f_v^m f_v^{-m} = 1$. $\square$

8.3 Sliding permutations are quasi-isometries

Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space. For a positive real number $C > 0$, a map $f : \mathcal{X} \to \mathcal{X}$ is called a $C-$*quasi-isometry* if

$$\frac{1}{C} d_{\mathcal{X}}(x,y) - C \le d_{\mathcal{X}}(f(x), f(y)) \le C d_{\mathcal{X}}(x,y) + C,$$

for every points $x, y \in \mathcal{X}$, and $\sup_{z \in \mathcal{X}} d_{\mathcal{X}}(z, f(\mathcal{X})) < \infty$. In general, a map is called a *quasi-isometry* if it is a $C-$quasi-isometry for some $C > 0$. Our goal of this section is that sliding permutations defined in Section 8.2 are quasi-isometries.

Theorem 8.3.1. *Let $\mathcal{F}$ be a finite set of simple closed geodesics, and let $\tilde{X}$ be the dual cube complex of $\mathcal{F}$ with the combinatorial metric d. If γ is a simple closed geodesic of $\mathcal{F}$ and v is a vertex on the dual tree $\tilde{X}_\gamma$ of γ, then the sliding permutation f_v centered at v is a $\left(1 + 3\sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)\right)$–quasi-isometry on $(\tilde{X}, d)$.*

Before we achieve our goal, we need some lemmas.

Lemma 8.3.2. *If a hyperplane $\hat{\mathfrak{h}}$ of $\tilde{X}$ crosses three hyperplanes $\hat{\mathfrak{h}}_0$, $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ which are pairwise disjoint, then some hyperplane $\hat{\mathfrak{h}}_i$ separates $\hat{\mathfrak{h}}_{i+1}$ from $\hat{\mathfrak{h}}_{i+2}$. (i, $i+1$ and $i+2$ are cyclically ordered.)*

Proof. Let h be the right-handed generator of $\hat{\mathfrak{h}}$. For each i, some halfspace $\mathfrak{h}_i$ of $\hat{\mathfrak{h}}_i$ is contained in $\mathcal{GH}(h)$ by Lemma 7.2.6.(2). By the hypothesis and Lemma 7.2.5, it holds, by re-ordering halfspaces, that $\mathfrak{h}_1 \supsetneq \mathfrak{h}_2 \supsetneq \mathfrak{h}_3$. Hence, $\hat{\mathfrak{h}}_2$ separates $\mathfrak{h}_1$ from $\mathfrak{h}_3$. $\square$

Lemma 8.3.3. *Let γ be a simple closed geodesic of $\mathcal{F}$. If $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}'$ are distinct hyperplanes of $\tilde{X}_\gamma$, then the number of hyperplanes of $\tilde{X}$ which cross both $\pi_\gamma^{-1}(\hat{\mathfrak{h}})$ and $\pi_\gamma^{-1}(\hat{\mathfrak{h}}')$ is at most $\sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)$.*

Proof. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be the disjoint halfspaces of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}'$, respectively. If h is the right-handed generator of $\mathfrak{h}$ and n is a nonzero integer, then we have $d_\gamma(\mathfrak{h}, \mathfrak{h}') = d_\gamma(\mathfrak{h}, h^n\mathfrak{h}')$. So neither $\mathfrak{h}' \supsetneq h^n\mathfrak{h}'$ nor $h^n\mathfrak{h}' \supsetneq \mathfrak{h}'$. Then the contraposition of Lemma 8.3.2 implies that, for all nonzero integer n, there is no hyperplane of $\tilde{X}$ which crosses $\pi_\gamma^{-1}(\hat{\mathfrak{h}})$, $\pi_\gamma^{-1}(\hat{\mathfrak{h}}')$ and $h^n\pi_\gamma^{-1}(\hat{\mathfrak{h}}')$ simultaneously.

Let $\mathbf{A}$ be the subset of $\mathcal{GH}(h) = \mathcal{GH}(h, \mathcal{F})$ satisfying that $\mathfrak{k} \in \mathbf{A}$ if and only if the hyperplane $\hat{\mathfrak{k}}$ of $\mathfrak{k}$ crosses both $\pi_\gamma^{-1}(\hat{\mathfrak{h}})$ and $\pi_\gamma^{-1}(\hat{\mathfrak{h}}')$. For every halfspace $\mathfrak{k}$ of $\mathbf{A}$ and a nonzero integer n, since $h^n\hat{\mathfrak{k}}$ crosses $\pi_\gamma^{-1}(\hat{\mathfrak{h}})$ and $h^n\pi_\gamma^{-1}(\hat{\mathfrak{h}}')$, it is satisfied that $h^n\hat{\mathfrak{k}}$ does not cross $\pi_\gamma(\hat{\mathfrak{h}}')$ by the above.

By the previous paragraph, the map $\mathbf{A} \to \langle h\rangle\backslash\mathcal{GH}(h)$ defined by $\mathfrak{k} \mapsto \langle h\rangle\mathfrak{k}$ is injective. By Proposition 7.2.7 and Proposition 7.3.3, $|\mathbf{A}|$ is at most the translation length of h, which is $\sum_{\alpha\in\mathcal{F}} i(\alpha, \gamma)$. □

Remark 8.3.4. By Proposition 6.3.13, Lemma 8.3.3 induces that, for every distinct hyperplanes $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}'$ of $\tilde{X}_\gamma$, the number of horizontal hyperplanes of the bridge $B(\pi_\gamma^{-1}(\hat{\mathfrak{H}}), \pi_\gamma^{-1}(\hat{\mathfrak{H}}'))$ is at most $\sum_{\alpha\in\mathcal{F}} i(\alpha, \gamma)$.

Lemma 8.3.5. *Let γ be a simple closed geodesic of $\mathcal{F}$, and let $\mathfrak{h}$ be a halfspace of $\tilde{X}_\gamma$ with the right-handed generator h. If $x \in \pi_\gamma^{-1}(\mathfrak{h}^*)$ and $y \in \pi_\gamma^{-1}(\mathfrak{h})$, then*

$$d(x, hy) \leq d(x, y) + \sum_{\alpha\in\mathcal{F}} i(\alpha, \gamma)$$

Proof. Let $x' := \pi_{\mathcal{N}(\hat{\mathfrak{H}})}(x)$ and $y' := \pi_{\mathcal{N}(\hat{\mathfrak{H}})}(y)$. Then $d(x, y) = d(x, x') + d(x', y') + d(y', y)$ by Lemma 6.3.11. Since $hy' = \pi_{\mathcal{N}(\hat{\mathfrak{H}})}(hy)$ and $d(y', hy') = \mathrm{tr}(h) \leq \sum_{\alpha\in\mathcal{F}} i(\alpha, \gamma)$ by Lemma 8.3.3, we have

$$\begin{aligned} d(x, hy) &= d(x, x') + d(x', hy') + d(hy', hy) \\ &\leq d(x, x') + d(x', y') + d(y', hy') + d(y', y) \\ &\leq d(x, y) + \sum_{\alpha\in\mathcal{F}} i(\alpha, \gamma). \end{aligned}$$

□

Proof of Theorem 8.3.1. Since f_v is bijective on $\tilde{X}^{(0)}$, we need only to check two inequalities. Choose two vertices x and y in $\tilde{X}$. If $\pi_\gamma(x) = \pi_\gamma(y)$, then $d(f_v(x), f_v(y)) = d(x, y)$ by Lemma 8.1.5.

Assume that $v = \pi_\gamma(x)$ and $v \neq \pi_\gamma(y)$. Let $\mathcal{O}(v) \cap \mathcal{I}(\pi_\gamma(y)) = \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_n\}$, and let h_i be the right-handed generator of $\mathfrak{h}_i$ for each $i = 1, \ldots, n$. By

Lemma 6.3.11,

$$d(x,y) = d(x_0,y_0) + d(y_0,x_1) + d(x_1,y_2) + \cdots + d(y_{n-1},x_n) + d(x_n,y_n)$$

where

- $x_0 := x$,
- $x_n := \pi_{\mathcal{N}(\hat{\mathfrak{h}}_n)}(y)$,
- $y_0 := \pi_{\mathcal{N}(\hat{\mathfrak{h}}_1)}(x)$,
- $y_n := y$, and
- $x_i := \pi_{B(\hat{\mathfrak{h}}_i,\hat{\mathfrak{h}}_{i+1})}(x)$ and $y_i := \pi_{\mathcal{N}(\hat{\mathfrak{h}}_{i+1})}(x)$ for each $i = 1,\dots,n-1$.

See Figure 6.2. On the other hand, since $\mathcal{O}(v) \cap \mathcal{I}(f_v(\pi_\gamma(y))) = \{\mathfrak{h}_1 \supsetneq h_1\mathfrak{h}_2 \supsetneq \cdots \supsetneq h_1 \dots h_{n-1}\mathfrak{h}_n\}$ by Lemma 8.2.1, we obtain another equation

$$d(x, f_v(y)) = d(x'_0,y'_0) + d(y'_0,x'_1) + d(x'_1,y'_2) + \cdots + d(y'_{n-1},x'_n) + d(x'_n,y'_n)$$

where

- $x'_0 := x$,
- $x'_n := \pi_{\mathcal{N}(h_1 \dots h_{n-1}\hat{\mathfrak{h}}_n)}(y)$,
- $y'_0 := \pi_{\mathcal{N}(\hat{\mathfrak{h}}_1)}(x)$,
- $y'_n := f_v(y)$, and
- $x'_i := \pi_{B(h_1\dots h_{i-1}\hat{\mathfrak{h}}_i, h_1\dots h_i\hat{\mathfrak{h}}_{i+1})}(x)$ and $y'_i := \pi_{\mathcal{N}(h_1\dots h_i\hat{\mathfrak{h}}_{i+1})}(x)$ for each $i = 1,\dots,n-1$.

Then $d(x'_0,y'_0) = d(x_0,y_0)$ by Lemma 8.1.5 and

$$\begin{aligned} d(x'_n,y'_n) &= d(\pi_{h_1\dots h_n\mathcal{N}(\hat{\mathfrak{H}}_n)}(h_1\dots h_n y), h_1\dots h_n y) \\ &= d(\pi_{\mathcal{N}(\hat{\mathfrak{H}}_n)}(y), y) = d(x_n,y_n). \end{aligned}$$

For each $i = 1,\dots,n-1$, the distance between x'_i and y'_i is the number of vertical hyperplanes of $B(h_1\dots h_i\hat{\mathfrak{h}}_i, h_1\dots h_i\hat{\mathfrak{h}}_{i+1})$ because $y'_i = \pi_{\mathcal{N}(h_1\dots h_i\hat{\mathfrak{h}}_{i+1})}(x'_i)$.

Note that $B(h_1 \dots h_i \hat{\mathfrak{h}}_i, h_1 \dots h_i \hat{\mathfrak{h}}_{i+1})$ is equal to $h_1 \dots h_i B(\hat{\mathfrak{h}}_i, \hat{\mathfrak{h}}_{i+1})$. Since $h_1 \dots h_i$ is an isometry, we have $d(x'_i, y'_i) = d(x_i, y_i)$ for each $i = 1, \dots, n-1$.

For $i = 1, \dots n-1$, two vertices y'_i and $h_1 \dots h_i y_i$ are contained in the intersection of $h_1 \dots h_i B(\hat{\mathfrak{h}}_i, \hat{\mathfrak{h}}_{i+1})$ and $h_1 \dots h_i \mathcal{N}(\hat{\mathfrak{h}}_{i+1})$. So every hyperplane separating $h_1 \dots h_i y_i$ from y'_i is a horizontal hyperplane of $h_1 \dots h_i B(\hat{\mathfrak{h}}_i, \hat{\mathfrak{h}}_{i+1})$. Then $d(h_1 \dots h_i y_i, y'_i)$ is at most the number of horizontal hyperplanes of $h_1 \dots h_i B(\hat{\mathfrak{h}}_i, \hat{\mathfrak{h}}_{i+1})$, which is $\sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)$ by Remark 8.3.4. Similarly, we have $d(h_1 \dots h_i x_i, x'_i) \leq \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)$ for each $i = 1, \dots, n-1$. For each $i = 0, \dots, n-1$, because of Lemma 8.3.5 and the fact that $d(y_i, x_{i+1}) \geq 1$, it holds that

$$\begin{aligned} d(y'_i, x'_{i+1}) &\leq d(y'_i, h_1 \dots h_i y_i) + d(h_1 \dots h_i y_i, h_1 \dots h_{i+1} x_{i+1}) \\ &\qquad\qquad\qquad\qquad + d(h_1, \dots h_{i+1} x_{i+1}, x'_{i+1}) \\ &\leq d(y_i, h_{i+1} x_{i+1}) + 2 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma) \\ &\leq d(y_i, x_{i+1}) + 3 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma) \\ &\leq \left(1 + 3 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)\right) d(y_i, x_{i+1}). \end{aligned}$$

If we change the roles $\{x_l, y_l\}$ and $\{x'_l, y'_l\}$ and apply f_v^{-1} instead of f_v, then we obtain that

$$d(y_l, x_{l+1}) \leq \left(1 + 3 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)\right) d(y'_i, x'_{i+1})$$

for each $i = 0, \dots, n-1$, in a similar way. So

$$\begin{aligned} d(x, f_v(y)) &= \sum_{i=0}^{n} d(x'_i, y'_i) + \sum_{i=0}^{n-1} d(y'_i, x'_{i+1}) \\ &\leq \sum_{i=0}^{n} d(x_i, y_i) + \sum_{i=0}^{n-1} \left(1 + 3 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)\right) d(y_i, x_{i+1}) \\ &\leq \left(1 + 3 \sum_{\alpha \in \mathcal{F}} i(\alpha, \gamma)\right) d(x, y) \end{aligned}$$

and

$$\begin{aligned} d(x, f_v(y)) &= \sum_{i=0}^{n} d(x'_i, y'_i) + \sum_{i=0}^{n-1} d(y'_i, x'_{i+1}) \\ &\geq \sum_{i=0}^{n} d(x_i, y_i) + \sum_{i=0}^{n-1} \frac{1}{1 + 3\sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)} d(y_i, x_{i+1}) \\ &\geq \frac{1}{1 + 3\sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)} d(x, y). \end{aligned}$$

For arbitrary vertices x and y, if $u := \pi_\gamma(x)$, let g denote $f_v f_u^{-1}$, which is an element of $\pi_1(\Sigma)$ by Proposition 8.2.5.(2). Then

$$\begin{aligned} d(f_v(x), f_v(y)) &= d(f_v f_u^{-1} f_u(x), f_v f_u^{-1} f_u(y)) \\ &= d(g f_u(x), g f_u(y)) \\ &= d(x, f_u(y)) \end{aligned}$$

By the above,

$$\frac{1}{1 + 3\sum_{\alpha} i(\alpha,\gamma)} d(x, y) \leq d(f_v(x), f_v(y)) \leq \left(1 + 3\sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)\right) d(x, y).$$

Therefore, f_v is a $\left(1 + 3\sum_{\alpha\in\mathcal{F}} i(\alpha,\gamma)\right)$–quasi-isometry. □

If a *$\mathcal{QI}$-relation* $\sim_{\mathcal{QI}}$ is the equivalence relation on the set of all quasi-isometries on a metric space $\mathcal{X}$, defined by

$$f_1 \sim_{\mathcal{QI}} f_2 \quad \Longleftrightarrow \quad \sup_{p\in\mathcal{X}} d(f_1(p), f_2(p)) < \infty,$$

then the quotient of the set of all quasi-isometries of $\mathcal{X}$ by $\sim_{\mathcal{QI}}$, denoted by $\mathcal{QI}(\mathcal{X})$ is a group with the composition. (cf. Chapter I.8 in [7].)

Note that every isometry is a quasi-isometry. So the map $\mathrm{Isom}(\mathcal{X}) \to \mathcal{QI}(\mathcal{X})$ sending each isometry to its equivalence class is a homomorphism. In general, this map is not injective. But it does not happen in our case by next lemma.

Lemma 8.3.6. *The homomorphism $\pi_1(\Sigma) \to \mathcal{QI}(\tilde{X})$, $g \mapsto \bar{g}$ is injective.*

Proof. Choose a nonidentity element $g \in \pi_1(\Sigma)$. First, for each simple closed geodesic $\gamma \in \mathcal{F}$, we will show that $\sup_{v \in \tilde{X}_\gamma} d_\gamma(v, gv) = \infty$. If the translation length of g on $\tilde{X}_\gamma$ is nonzero, let L be the axis of g. For each positive integer N, let u be a vertex of $\tilde{X}_\gamma$ satisfying that $d(u, L) > N$. If π_L is the collapsing to L, then we have $d_\gamma(u, gu) = d_\gamma(u, \pi_L(u)) + d_\gamma(\pi_L(u), \pi_L(gu)) + d_\gamma(\pi_L(gu), gu) > 2N$.

When the translation length of g on $\tilde{X}_\gamma$ is zero, let v be a vertex which is fixed by g. Let $\mathrm{Fix}(g)$ be the subtree of $\tilde{X}$ which is fixed by g pointwise. In fact, $\mathrm{Fix}(g)$ is either a vertex or an edge in our case. Let $\pi_{\mathrm{Fix}(g)}$ be the collapsing to $\mathrm{Fix}(g)$. For every positive integer N, if u is a vertex such that $d_\gamma(u, \mathrm{Fix}(g)) > N$, then we have $d_\gamma(u, gu) = d_\gamma(u, \pi_{\mathrm{Fix}(g)}(u)) + d_\gamma(\pi_{\mathrm{Fix}(g)}(gu), gu) > 2N$. So g is not $\mathcal{QI}$-related to the identity with respect to $\tilde{X}_\gamma$.

For every vertex x of $\tilde{X}$, we have $d(x, gx) \geq d_\gamma(\pi_\gamma(x), \pi_\gamma(gx))$ for every $\gamma \in \mathcal{F}$, (by Lemma 7.1.3.) Therefore, it holds that $\sum_{x \in \tilde{X}} d(x, gx) = \infty$. $\square$

By the previous lemma, we can consider $\pi_1(\Sigma)$ as a subgroup of $\mathcal{QI}(\tilde{X})$. Let $\mathcal{N}(\pi_1(\Sigma)) = \mathcal{N}_{\mathcal{QI}(\tilde{X})}(\pi_1(\Sigma))$ denote the normalizer of $\pi_1(\Sigma)$ in the quasi-isometry group $\mathcal{QI}(\tilde{X})$. Then, by Proposition 8.2.5.(3), for every simple closed geodesic $\gamma \in \mathcal{F}$ and a vertex $v \in \tilde{X}_\gamma$, the equivalence class of the sliding permutation f_v, denoted by $\bar{f}_v$, is contained in $\mathcal{N}(G)$. It induces that $\pi_1(\Sigma)$ is a proper subgroup of $\mathcal{N}(\pi_1(\Sigma))$.

Proposition 8.3.7. *For every sliding permutation f_v on $\tilde{X}$, the equivalence class $\bar{f}_v \in \mathcal{QI}(\tilde{X})$ of f_v is contained in the normalizer of $\pi_1(\Sigma)$.*

Lemma 8.3.8. *The conjugate action of $\mathcal{N}(\pi_1(\Sigma))$ on $\pi_1(\Sigma)$ is faithful.*

Proof. Let $\bar{f}$ be an equivalence class in $\mathcal{N}(\pi_1(\Sigma))$ such that $g = \bar{f} g \bar{f}^{-1}$ for all $g \in \pi_1(\Sigma)$. Choose a quasi-isometry f of $\bar{f}$. By Proposition 7.1.6, there is a compact subcomplex Y in $\tilde{X}$ such that $\bigcup_{g \in \pi_1(\Sigma)} gY = \tilde{X}$. So, for each vertex x of $\tilde{X}$, there is an element g_x of $\pi_1(\Sigma)$ such that $g_x x \in Y$. Note that $\{f(y) \,|\, y \in Y^{(0)}\}$ is bounded since f is a quasi-isometry. So $d(x, fx) = d(g_x x, g_x f x) = d(g_x x, f g_x x) \leq \sup_{y \in Y} d(y, fy) < \infty$. Hence, $f \sim_{\mathcal{QI}} 1$. $\square$

We call the equivalence class of a sliding permutation a *sliding quasi-isometry.*

Remark 8.3.9. The fundamental group $\pi_1(\Sigma)$ is a subgroup of the automorphism group of $\tilde{X}$ by Lemma 7.1.9, moreover, it is considered as a subgroup of $\mathcal{QI}(\tilde{X})$. Because of Lemma 8.3.8, we regard the normalizer $\mathcal{N}(\pi_1(\Sigma))$ with the subgroup of the automorphism group of $\pi_1(\Sigma)$. In fact, we see the following diagram.

$$\begin{tikzcd}
\mathcal{QI}(\tilde{X}) \arrow[d, dash] & \\
\mathcal{N}(\pi_1(\Sigma)) \arrow[r, hook] \arrow[d, dash] & \mathrm{Aut}(\pi_1(\Sigma)) \arrow[d, dash] \\
\pi_1(\Sigma) \arrow[r, equal] & \mathrm{Inn}(\pi_1(\Sigma))
\end{tikzcd}$$

The normalizer of $\pi_1(\Sigma)$ contains a lot of sliding quasi-isometries. If $\mathcal{G}$ is a nonempty subset of $\mathcal{F}$, then all sliding quasi-isometries on $\mathcal{N}_{\mathcal{G}}(\pi_1(\Sigma))$ are contained in $\mathcal{N}_{\mathcal{F}}(\pi_1(\Sigma))$. That is, the bigger $\mathcal{F}$ is, the more sliding quasi-isometries are contained in $\mathcal{N}_{\mathcal{F}}(\pi_1(\Sigma))$.

The quotient group $\mathcal{N}(\pi_1(\Sigma))/\pi_1(\Sigma)$ is one of the major things in our consideration. By Lemma 8.2.8, for every simple closed geodesic γ of $\mathcal{F}$, all sliding quasi-isometries centered at vertices of $\tilde{X}_\gamma$ are contained in the same left coset of $\mathcal{N}(\pi_1(\Sigma))/\pi_1(\Sigma)$.

Definition 8.3.10. For each simple closed geodesic γ of $\mathcal{F}$, let $\bar{f}_\gamma$ denote the left coset of $\mathcal{N}(\pi_1(\Sigma))/\pi_1(\Sigma)$ which contains all sliding quasi-isometries centered at vertices of $\tilde{X}_\gamma$. We call $\bar{f}_\gamma$ the *sliding (left) coset* of γ.

Then Proposition 8.2.11 immediately implies that, whenever two simple closed geodesics α and β are disjoint, their sliding cosets commute each other.

Lemma 8.3.11. *If $\mathcal{F}$ is decomposed into $\mathcal{A}$ and $\mathcal{B}$ such that $i(\alpha,\beta) = 0$ for all $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, then $\langle\{\bar{f}_\gamma \,|\, \gamma \in \mathcal{F}\}\rangle$ is isomorphic to $\langle\{\bar{f}_\alpha \,|\, \alpha \in \mathcal{A}\}\rangle \times \langle\{\bar{f}_\beta \,|\, \beta \in \mathcal{B}\}\rangle$.*

Remark 8.3.12. For every simple closed geodesic $\gamma \in \mathcal{F}$, the two injective homomorphisms $\mathcal{N}(\pi_1(\Sigma))/\pi_1(\Sigma) \hookrightarrow \mathrm{Out}(\pi_1(\Sigma)) \hookleftarrow \mathrm{Mod}(\Sigma)$ sends both the sliding coset $\bar{f}_\gamma$ of γ and the Dehn twist T_γ along γ to the same outer automorphism.

Theorem 8.3.13. *If Σ is a closed orientable surface of genus ≥ 2, there exist finitely many simple closed curves such that the homomorphism from* $\mathrm{Aut}(\pi_1(\Sigma))$ *to the quasi-isometry group of the dual cube complex $\tilde{X}$ of these curves is injective and the following diagram is commutative.*

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[r, hook] \arrow[d, hook] & \mathrm{Isom}(\tilde{X}) \arrow[d] \\ \mathrm{Aut}(\pi_1(\Sigma)) \arrow[r, hook] & \mathcal{QI}(\tilde{X}) \end{tikzcd}$$

Proof. By the Dehn–Nielsen–Baer theorem, the following is a short exact sequence:

$$0 \longrightarrow \pi_1(\Sigma) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma)) \longrightarrow \mathrm{Mod}^{\pm}(\Sigma) \longrightarrow 0$$

where $\mathrm{Mod}^{\pm}(\Sigma)$ is the extended mapping class group of Σ (containing $\mathrm{Mod}(\Sigma)$ as an index 2 subgroup). Note that the image of $\pi_1(\Sigma)$ in the sequence is the group of inner automorphisms.

Let $\mathcal{F}$ denote a finite set of simple closed curves such that their Dehn twists generate $\mathrm{Mod}(\Sigma)$.[1] If $\tilde{T}$ is a lift of one of those Dehn twists, then there is a quasi-isometry f of the dual cube complex of $\mathcal{F}$ such that $(\tilde{T}g\tilde{T}^{-1})(v) = fgf^{-1}(v)$ by Theorem 8.2.6 and Theorem 8.3.1. That is, $\tilde{T}$ acts quasi-isometrically on the dual cube complex of $\mathcal{F}$.

For every automorphism ϕ of the surface group, the coset $\phi \cdot \mathrm{Inn}(\pi_1(\Sigma))$ can be decomposed into the multiplication of Dehn twists and the orientation-reversing map.[2] By multiplying the orientation-reversing map if needed, we assume that ϕ is orientation-preserving. Write $\phi \cdot \mathrm{Inn}(\pi_1(\Sigma))$ as a product of powers of Dehn twists $T_{\gamma_1}^{n_1} \dots T_{\gamma_m}^{n_m}$.

If $\tilde{T}_{\gamma_i}$ is a lift of T_{γ_i} for each $i \in \{1, \dots, m\}$, there is an element g_0 of $\pi_1(\Sigma)$ such that ϕ is the conjugate action of $\tilde{T}_{\gamma_1}^{n_1} \dots \tilde{T}_{\gamma_m}^{n_m} g_0$, that is,

$$\phi(g) = (\tilde{T}_{\gamma_1}^{n_1} \dots \tilde{T}_{\gamma_m}^{n_m} g_0) g (\tilde{T}_{\gamma_1}^{n_1} \dots \tilde{T}_{\gamma_m}^{n_m} g_0)^{-1}$$

for all $g \in \pi_1(\Sigma)$; see Section 2.6.1 for the relation between automorphisms and homeomorphisms on $\mathbb{H}^2$. Define the map from $\mathrm{Aut}(\pi_1(\Sigma))$ to the quasi-

[1] For example, consider a Humphries generator or a Lickorish generator.

[2] $\mathrm{Inn}(\pi_1(\Sigma))$ is the group of inner automorphisms of the surface group. It is the image of $\pi_1(\Sigma)$ in the above short exact sequence.

isometry group of the dual cube complex of $\mathcal{F}$ such that $\phi \mapsto \tilde{T}_{\gamma_1}^{n_1} \ldots \tilde{T}_{\gamma_m}^{n_m} g_0$ for every automorphism $\phi \in \mathrm{Aut}(\pi_1(\Sigma))$.

Then this is a well-defined injective homomorphism because of Lemma 8.3.8. This homomorphism satisfies that the restriction to $\mathrm{Inn}(\pi_1(\Sigma))$ is equal to the action of the isometric action of $\pi_1(\Sigma)$. $\square$

Appendix A

Proofs of two propositions

A.1 Proof of Proposition 3.1.3

Proposition A.1.1. *Let $R > 0$ be the collar length of a simple closed geodesic γ. For every $\epsilon \in (0, R)$, there is a $\pi_1(\Sigma)$-equivariant surjective continuous map $\Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \tilde{X}_\gamma$ such that $\Phi_{\gamma,\epsilon}(\tilde{\gamma}) = \tau(\tilde{\gamma})$ for each lift $\tilde{\gamma}$ of γ.*

Furthermore, if α is a simple closed geodesic which is distinct from γ, then there is $\epsilon_0 = \epsilon_0(\alpha, \gamma) > 0$ such that, for every lift $\tilde{\alpha}$ of α and every $0 < \epsilon < \min\{\epsilon_0, R\}$, the image $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ is either a geodesic or a vertex in $\tilde{X}_\gamma$.

To prove the proposition, we need first to flatten the neighborhood of each wall.

Lemma A.1.2. *Let H be a halfspace bounded by a geodesic $\tilde{\gamma}$ of $\mathbb{H}^2$. Then, for every point $q \in \tilde{\gamma}$ and $\epsilon > 0$, there is a homeomorphism $\phi_{\mathsf{H},q,\epsilon} : \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \to \mathbb{R} \times [0, 1]$ such that*

- $\phi_{\mathsf{H},q,\epsilon}(\mathsf{H} \cap \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}) = \mathbb{R} \times \left[\frac{1}{2}, 1\right]$ *with* $\phi_{\mathsf{H},q,\epsilon}(q) = \left(0, \frac{1}{2}\right)$ *and,*
- *for every generator $g \in \pi_1(\Sigma)$ of the stabilzer of $\tilde{\gamma}$, the induced map $\phi_{\mathsf{H},q,\epsilon} \circ g \circ \phi^{-1}_{\mathsf{H},q,\epsilon}$ on $\mathbb{R} \times [0, 1]$ is a translation of length 1.*

Proof. First, we consider the orientation of $\tilde{\gamma}$ which is given as the boundary of H. Let $\vec{\gamma}$ be the subray of $\tilde{\gamma}$ which starts from q_0 and follows the orientation of $\tilde{\gamma}$. Let $P_{\tilde{\gamma}} : \mathbb{H}^2 \to \tilde{\gamma}$ be the closest-point projection. We define two maps

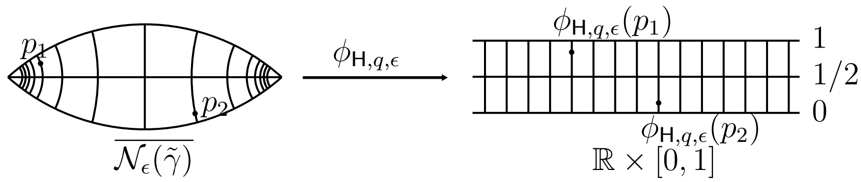

Figure A.1: The map $\phi_{\mathsf{H},q,\epsilon}$ between two strips is a homeomorphism.

$\lambda_1^{\mathsf{H},q} : \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \to \mathbb{R}$ and $\lambda_2^{\mathsf{H},\epsilon} : \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \to [0,1]$ by

$$\lambda_1^{\mathsf{H},q}(p) = \begin{cases} d_{\mathbb{H}^2}(q, P_{\tilde{\gamma}}p)/l(\gamma) & \text{if } P_{\tilde{\gamma}}p \in \vec{\gamma}, \\ -d_{\mathbb{H}^2}(q, P_{\tilde{\gamma}}p)/l(\gamma), & \text{otherwise, and} \end{cases}$$

$$\lambda_2^{\mathsf{H},\epsilon}(p) = \begin{cases} \frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p) + \frac{1}{2} & \text{if } p \in \mathsf{H} \cap \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}, \\ -\frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p) + \frac{1}{2}, & \text{otherwise,} \end{cases}$$

where $l(\gamma)$ is the length of γ. Define a map $\phi_{\mathsf{H},q,\epsilon} : \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \to \mathbb{R} \times [0,1]$ by $\phi_{\mathsf{H},q,\epsilon}(p) = (\lambda_1^{\mathsf{H},q}(p), \lambda_2^{\mathsf{H},\epsilon}(p))$. (See Figure A.1.) Then we can easily check that the properties in Lemma A.1.2 holds for $\phi_{\mathsf{H},q,\epsilon}$.

Let $d^1_{\mathbb{H}^2}$ be the metric on $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$ defined by $d^1_{\mathbb{H}^2}(p_1,p_2) = |\lambda_1^{\mathsf{H},q}(p_1) - \lambda_1^{\mathsf{H},q}(p_2)| + |\lambda_2^{\mathsf{H},\epsilon}(p_1) - \lambda_2^{\mathsf{H},\epsilon}(p_2)|$ for all $p_1, p_2 \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. If $d^1_{\mathbb{R}^2}$ is the metric on $\mathbb{R}\times[0,1]$ which is the subspace metric of the L^1-metric on $\mathbb{R}^2$, the map $\phi_{\mathsf{H},q,\epsilon}$ is an isometry between $(\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}, d^1_{\mathbb{H}^2})$ and $(\mathbb{R}\times[0,1], d^1_{\mathbb{R}^2})$. Note that the topology given by $d^1_{\mathbb{R}^2}$ is as same as the subspace topology on $\mathbb{R}\times[0,1]\subset\mathbb{R}^2$. To prove $\phi_{\mathsf{H},q,\epsilon}$ is a homeomorphism, we need only to show that the subspace topology on $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \subset \mathbb{H}^2$ is identical to the topology given by $d^1_{\mathbb{H}^2}$.

We fix a point $p \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$ and $r > 0$. Let

$$Q(p,r) = \{p' \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \,|\, d_{\mathbb{H}^2}(p,p') = r\}.$$

By the continuity of $P_{\tilde{\gamma}}$, the map $Q(p,r) \to \mathbb{R}$ defined by $p' \mapsto d^1_{\mathbb{H}^2}(p,p') = d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, P_{\tilde{\gamma}}p') + |\lambda_2^{\mathsf{H},\epsilon}(p) - \lambda_2^{\mathsf{H},\epsilon}(p')|$ is continuous with respect to the metric $d_{\mathbb{H}^2}$. And this map has always a positive value for all $p' \in Q(p,r)$ because $p' \neq p$. Since $Q(p,r)$ is compact, the map has a positive minimum $m_p = m_p(r)$ and a maximum $M_p = M_p(r)$.

It implies that, for each $p \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$, there are two increasing maps $m_p, M_p :$

$\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that, if $p' \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$ and $r := d_{\mathbb{H}^2}(p, p')$, then

$$m_p(r) \le d^1_{\mathbb{H}^2}(p, p') \le M_p(r).$$

So $B_{m_p(r)}(p; d^1_{\mathbb{H}^2}) \subseteq B_r(p; d_{\mathbb{H}^2}) \subseteq B_{M_p(r)}(p; d^1_{\mathbb{H}^2})$ where they are open balls centered at p with respect to $d_{\mathbb{H}^2}$ and $d^1_{\mathbb{H}^2}$, respectively. Hence, the topologies given by $d_{\mathbb{H}^2}$ and $d^1_{\mathbb{H}^2}$ are equivalent. □

Definition A.1.3. For each halfspace $\mathfrak{h}$ of $\tilde{X}_\gamma$, let $i_{\mathfrak{h}}$ be the orientation-preserving isometry from $[0, 1]$ to the oriented edge whose terminal halfspace is $\mathfrak{h}$. Then, for each halfspace H bounded by a geodesic $\tilde{\gamma}$ of $\mathbb{H}^2$, we define a map $\psi_{\mathsf{H},\epsilon} : \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \to e$ as the composition of

$$\overline{\mathcal{N}_\epsilon(\tilde{\gamma})} \xrightarrow{\phi_{\mathsf{H},q,\epsilon}} \mathbb{R} \times [0, 1] \twoheadrightarrow [0, 1] \xrightarrow{i_{\tau(\mathsf{H})}} e \ .$$

Concretely, $\psi_{\mathsf{H},\epsilon} = i_{\tau(\mathsf{H})} \circ \lambda_2^{\mathsf{H},\epsilon}$ where $\lambda_2^{\mathsf{H},\epsilon}$ is the map defined in the proof of Lemma A.1.2.

Note that $\psi_{\mathsf{H},\epsilon}$ does not depend on the choice of a point q for $\phi_{\mathsf{H},q,\epsilon}$.

Lemma A.1.4. *1. We have $\psi_{\mathsf{H},\epsilon} = \psi_{\mathsf{H}^*,\epsilon}$ for all $\mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2}$. So we will use the notation $\psi_{\tilde{\gamma},\epsilon}$ instead of $\psi_{\mathsf{H},\epsilon}$.*

2. For all $g \in \pi_1(\Sigma)$ and $\tilde{\gamma} \in \mathcal{L}_\gamma$, if e is the edge whose midpoint is $\tau(\tilde{\gamma})$, and if $g \restriction_e$ and $g \restriction_{\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}}$ are the restriction of $\pi_1(\Sigma)$ to e and $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$, respectively, then $(g \restriction_e)\psi_{\tilde{\gamma},\epsilon} = \psi_{g\tilde{\gamma},\epsilon}(g \restriction_{\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}})$. That is, the following diagram is commutative.

$$\begin{array}{ccc} \overline{\mathcal{N}_\epsilon(\tilde{\gamma})} & \xrightarrow{g} & \overline{\mathcal{N}_\epsilon(g\tilde{\gamma})} \\ \downarrow{\scriptstyle \psi_{\tilde{\gamma},\epsilon}} & & \downarrow{\scriptstyle \psi_{g\tilde{\gamma},\epsilon}} \\ e & \xrightarrow{g} & ge \end{array}$$

Proof. (1) Choose a point p in $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. If a halfspace H contains p, then

$$\psi_{\mathsf{H},\epsilon}(p) = i_{\tau(\mathsf{H})}\left(\frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p) + \frac{1}{2}\right) \text{ and}$$
$$\psi_{\mathsf{H}^*,\epsilon}(p) = i_{\tau(\mathsf{H}^*)}\left(-\frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p) + \frac{1}{2}\right).$$

Since $i_{\tau(\mathsf{H})}(1) = i_{\tau(\mathsf{H}^*)}(0)$, we have

$$d_\gamma(i_{\tau(\mathsf{H})}(1), \psi_{\mathsf{H},\epsilon}(p)) = \frac{1}{2} - \frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p) = d_\gamma(i_{\tau(\mathsf{H}^*)}(0), \psi_{\mathsf{H}^*,\epsilon}(p)).$$

So $\psi_{\mathsf{H},\epsilon}(p) = \psi_{\mathsf{H}^*,\epsilon}(p)$. The case that $p \in \mathsf{H}^*$ can be proved similarly.

(2) Choose a point $p \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. Let H be a halfspace which is bounded by $\tilde{\gamma}$ and contains p. If v is the endpoint of e which is contained in $\tau(\mathsf{H})$, then $d_\gamma(v, \psi_{\tilde{\gamma},\epsilon}(p)) = \frac{1}{2} - \frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{\tilde{\gamma}}p, p)$. On the other hand, note that $gp \in \overline{\mathcal{N}_\epsilon(g\tilde{\gamma})}$ and $gv \in g\tau(\mathsf{H}) = \tau(g\mathsf{H})$. Since $gp \in \mathsf{g}\mathsf{H}$ and gv is an endpoint of ge, we have $d^{\mathbb{R}}_\gamma(gv, \psi_{g\tilde{\gamma},\epsilon}(gp)) = \frac{1}{2} - \frac{1}{2\epsilon} d_{\mathbb{H}^2}(P_{g\tilde{\gamma}}gp, gp)$. Because $P_{g\tilde{\gamma}}gp = gP_{\tilde{\gamma}}p$,

$$d^{\mathbb{R}}_\gamma(gv, \psi_{g\tilde{\gamma},\epsilon}(gp)) = d^{\mathbb{R}}_\gamma(gv, g\psi_{\tilde{\gamma},\epsilon}(p)).$$

Therefore, $\psi_{g\tilde{\gamma},\epsilon}(gp) = g\psi_{\tilde{\gamma},\epsilon}(p)$ for all $p \in \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. □

Proof of Proposition A.1.1. Because ϵ is sufficiently small, $\bigsqcup_{\tilde{\gamma}\in\mathcal{L}_\gamma} \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$ is a disjoint union. For each $p \in \mathbb{H}^2$, define $\Phi_{\gamma,\epsilon}(p)$ as follows: if p lies in some $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$, then $\Phi_{\gamma,\epsilon}(p)$ is equal to $\psi_{\tilde{\gamma},\epsilon}(p)$; otherwise, $\Phi_{\gamma,\epsilon}(p)$ is the vertex of $\tilde{X}_\gamma$ whose inward orientation is $\{\tau(\mathsf{H}) \,|\, \mathsf{H} \in \mathcal{H}^{\mathbb{H}^2}_\gamma, p \in \mathsf{H}\}$. We claim that the map $\Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \tilde{X}_\gamma$ is what we want.

We need to check the continuity of $\Phi_{\gamma,\epsilon}$. Because $\psi_{\tilde{\gamma},\epsilon}$ is continuous for all $\tilde{\gamma}$, it is enough to show that $\Phi_{\gamma,\epsilon}$ is continuous near the boundary of $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$ for each $\tilde{\gamma} \in \mathcal{L}_\gamma$. Let p be a point on the boundary of some $\overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. If e is the image of $\psi_{\tilde{\gamma},\epsilon}$, then $\psi_{\tilde{\gamma},\epsilon}(p)$ is the endpoint of e so it is a vertex. If H is the halfspace of $\tilde{\gamma}$ which contains p, then we obtain the fact that $\psi_{\tilde{\gamma},\epsilon}(p)$ is contained in $\tau(\mathsf{H})$, by calculation. So the inward orientation of $\psi_{\tilde{\gamma},\epsilon}(p)$ is $\{\mathfrak{h} \,|\, e \subset \mathfrak{h}\} \cup \{\tau(\mathsf{H})\} = \{\tau(\mathsf{H}') \,|\, p \in \mathsf{H}', \mathsf{H}' \in \mathcal{H}^{\mathbb{H}^2}_\gamma\}$. Hence, $\Phi_{\gamma,\epsilon}$ is continuous at p.

The $\pi_1(\Sigma)$-equivariance of $\Phi_{\gamma,\epsilon}$ is followed by Lemma A.1.4.(2) and the $\pi_1(\Sigma)$-equivariance of τ. And, because the image of $\Phi_{\gamma,\epsilon}$ contains all edges of $\tilde{X}_\gamma$, it holds that $\Phi_{\gamma,\epsilon}$ is surjective.

If a simple closed curve α is disjoint from γ, then there is $\epsilon_0 > 0$ such that the ϵ_0-neighborhood of γ is also disjoint to α by the collar lemma. So, for every $0 < \epsilon < \epsilon_0$, every lift $\tilde{\alpha}$ of α does not intersect $\bigsqcup_{\tilde{\gamma}\in\mathcal{L}_\gamma} \overline{\mathcal{N}_\epsilon(\tilde{\gamma})}$. So $\Phi_{j,\epsilon}(\tilde{\alpha})$ is a vertex for all $0 < \epsilon < \min\{\epsilon_0, R\}$ and all lift $\tilde{\gamma}$ of α.

Now we consider the case that a simple closed geodesic α intersects γ. We claim that there is $\epsilon_0 = \epsilon_0(\gamma, \alpha) > 0$ such that, for every $0 < \epsilon < \epsilon_0$, a lift $\tilde{\gamma}$ intersects another lift $\tilde{\alpha}$ of α if and only if $\tilde{\alpha}$ intersects $\mathcal{N}_\epsilon(\tilde{\gamma})$.

Fix a lift $\tilde{\alpha}$ of α and suppose that the claim is false. In other words, for every $\epsilon > 0$, there is a lift $\tilde{\gamma}$ such that $\tilde{\alpha}$ intersects $\mathcal{N}_\epsilon(\tilde{\gamma})$ but it is disjoint from $\tilde{\gamma}$. Then, for each $n \in \mathbb{N}$, choose a lift $\tilde{\gamma}_n$ of γ such that $\tilde{\alpha} \cap \tilde{\gamma}_n = \emptyset$ and $\tilde{\alpha} \cap \mathcal{N}_{\frac{1}{n}}(\tilde{\gamma}_n) \neq \emptyset$. For each $n \in \mathbb{N}$, let q_n be a point on $\tilde{\alpha}$ which is closest to $\tilde{\gamma}_n$. Because α is compact, there is a subsequence $\{q_{n_l}\} \subseteq \{q_n\}$ such that $\xi(q_{n_l})$ converges to some point $\xi(q)$ as $l \to \infty$. (Note that ξ is the covering map from $\mathbb{H}^2$ to Σ.) If q is a point in the preimage of $\xi(q)$, then, for each l, let q'_{n_l} be a point on $\mathbb{H}^2$ such that q'_{n_l} is in the preimage of $\xi(q_{n_l})$ and the sequence q'_{n_l} converges to $\tilde{q}$ as $l \to \infty$. Since $q'_{n_l} = g_l q_{n_l}$ for some $g_l \in \mathrm{Stab}_G(\tilde{\alpha})$, there is a lift $\tilde{\gamma}'_{n_l}$ $(= g_l \tilde{\gamma}_{n_l})$ such that $\mathcal{N}_{\frac{1}{n_l}}(\tilde{\gamma}'_{n_l})$ contains q'_{n_l}. Then, for each l,

$$\begin{aligned} d_{\mathbb{H}^2}(\tilde{\gamma}'_{n_l}, \tilde{\gamma}'_{n_{l+1}}) &\leq d_{\mathbb{H}^2}(\tilde{\gamma}'_{n_l}, q'_{n_l}) + d_{\mathbb{H}^2}(q'_{n_l}, q'_{n_{l+1}}) + d_{\mathbb{H}^2}(q'_{n_{l+1}}, \tilde{\gamma}'_{n_{l+1}}) \\ &< \frac{1}{n_l} + \frac{1}{n_{l+1}} + d_{\mathbb{H}^2}(q'_{n_l}, q'_{n_{l+1}}). \end{aligned}$$

So $d_{\mathbb{H}^2}(\tilde{\gamma}'_{n_l}, \tilde{\gamma}'_{n_{l+1}}) \to 0$ as $l \to \infty$. Because the distance between two disjoint lifts of γ is more than the collar length of γ, there exists some $l_0 > 0$ satisfying that $\tilde{\gamma}'_{n_l} = \tilde{\gamma}'_{n_{l_0}}$ for all $l \geq l_0$. Then $q'_{n_l} = q$ for all $l \geq l_0$, and the point q is contained in $\mathcal{N}_{\frac{1}{n_l}}(\tilde{\gamma}'_{n_{l_0}})$ for all $l \geq l_0$. So $q \in \tilde{\gamma}'_{n_{l_0}}$, which implies that $\tilde{\gamma}'_{n_{l_0}}$ intersects $\tilde{\alpha}$. It is a contradiction. So the claim is proved.

Assume that ϵ is small enough to satisfy that a lift $\tilde{\gamma}$ intersects $\tilde{\alpha}$ transversally if and only if $\tilde{\alpha}$ intersects $\mathcal{N}_\epsilon(\tilde{\gamma})$. Let $\rho : (-\infty, \infty) \to \mathbb{H}^2$ be a geodesic path whose image is $\tilde{\alpha}$. Let $\{t_n\}_{n \in \mathbb{Z}} = \{t \in (-\infty, \infty) \,|\, \rho(t) \in \tilde{\gamma}$ for some lift $\tilde{\gamma}\}$ be an increasing sequence. For each $n \in \mathbb{Z}$, let $\tilde{\gamma}_n$ be the lift of γ where $\rho(t_n)$ lies. For each n, the distance between $\rho(t)$ and $\tilde{\gamma}_n$ increases strictly to the infinity as t goes farther away from t_n. So $I_n = \rho^{-1}(\mathcal{N}_\epsilon(\tilde{\gamma}_n))$ is a bounded open interval. Because ϵ is small, the closures $\bar{I}_n$ and $\bar{I}_m$ are disjoint whenever $n \neq m$.

Note that the image of $\Phi_{\gamma,\epsilon} \circ \rho$ is $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ and is injective on each interval I_n. If $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ is not a geodesic of $\tilde{X}_\gamma$, then there are distinct n and m such that $(\Phi_{\gamma,\epsilon} \circ \rho)(I_n) = (\Phi_{\gamma,\epsilon} \circ \rho)(I_m)$. If $\tilde{\gamma}$ is the lift intersecting $(\Phi_{\gamma,\epsilon} \circ \rho)(I_n)$, then $\tilde{\gamma}$ and $\tilde{\alpha}$ form a bigon, which is a contradiction. Therefore, $\Phi_{\gamma,\epsilon}(\tilde{\alpha})$ is a geodesic. $\square$

A.2 Proof of Theorem 8.2.6

This chapter is a sequel of Chapter A.1. We use the notations (such as $\phi_{\mathsf{H},q,\epsilon}$, $\psi_{\tilde{\gamma},\epsilon}$ and $\Phi_{\gamma,\epsilon}$) defined in Chapter A.1. For a simple closed geodesic γ of Σ and a sufficiently small $\epsilon > 0$, the map $\Phi_{\gamma,\epsilon} : \mathbb{H}^2 \to \tilde{X}_\gamma$ is the $\pi_1(\Sigma)$-equivariant surjective map defined in Proposition A.1.1.

Definition A.2.1 (Dehn twists)**.** Let $T : (\mathbb{R}/\mathbb{Z}) \times [0,1] \to (\mathbb{R}/\mathbb{Z}) \times [0,1]$ be defined by $T([s],t) = ([s+t],t)$ for all $([s],t) \in (\mathbb{R}/\mathbb{Z}) \times [0,1]$. Note that, if ϵ is very small, the ϵ-neighborhood of γ is homeomorphic to an annulus. So, if ϵ is small and $\mathcal{N}_\epsilon(\gamma)$ is the open ϵ-neighborhood of γ in Σ, the map $\phi_{\gamma,\epsilon} : \mathcal{N}_\epsilon(\gamma) \to (\mathbb{R}/\mathbb{Z}) \times [0,1]$, satisfying that the following diagram is commutative for all halfspace H bounded by a lift $\tilde{\gamma}$ of γ and a point $q \in \tilde{\gamma}$, is a well-defined homeomorphism.

$$\begin{tikzcd}
\mathcal{N}_\epsilon(\tilde{\gamma}) \arrow[r, "\phi_{\mathsf{H},q,\epsilon}"] \arrow[d] & \mathbb{R} \times [0,1] \arrow[d] \\
\mathcal{N}_\epsilon(\gamma) \arrow[r, dashed, "\phi_{\gamma,\epsilon}"] & (\mathbb{R}/\mathbb{Z}) \times [0,1]
\end{tikzcd}$$

Let $T_{\gamma,\epsilon}$ be the self-homeomorphism of Σ defined by

$$T_{\gamma,\epsilon}(p) = \begin{cases} p & \text{if } p \in \Sigma \setminus \mathcal{N}_\epsilon(\gamma), \\ (\phi_{\gamma,\epsilon}^{-1} \circ T \circ \phi_{\gamma,\epsilon})(p) & \text{if } p \in \mathcal{N}_\epsilon(\gamma). \end{cases}$$

Then $T_{\gamma,\epsilon}$ is a *Dehn twist* along γ. (Compare with the definition of Dehn twists in [17, Chapter 3].)

Our next theorem says that a sliding permutation on $\tilde{X}_\gamma$ can be realized by a lift of a Dehn twist.

Theorem A.2.2. *For a simple closed geodesic γ, let ϵ be a positive real number smaller than the constant of the collar lemma for γ. Then, for each vertex $v \in \tilde{X}_\gamma$, there is a lift $\tilde{T}_{v,\epsilon}$ of the Dehn twist $T_{\gamma,\epsilon}$ such that $\Phi_{\gamma,\epsilon} \circ \tilde{T}_{v,\epsilon} = f_v \circ \Phi_{\gamma,\epsilon}$.*

Proof. Choose a point q of $\mathbb{H}^2$ such that q does not lie on the closed ϵ-neighborhood of any lifts of γ and $\mathcal{I}(v) = \{\tau(\mathsf{H}) \,|\, q \in \mathsf{H} \text{ and } \mathsf{H} \in \mathcal{H}_\gamma^{\mathbb{H}^2}\}$. Then the image of q on Σ locates outside the support of $T_{\gamma,\epsilon}$. So there is a lift $\tilde{T}_{v,\epsilon}$ of

$T_{\gamma,\epsilon}$ which fixes q. Let another point p outside the ϵ-neighborhoods of all lifts of γ be given. Then $\Phi_{\gamma,\epsilon}(p)$ is a vertex of $\tilde{X}_\gamma$. If $\mathcal{O}(v) \cap \mathcal{I}(\Phi_{\gamma,\epsilon}(p))$ is empty, then p and q lie on the same connected component of $\mathbb{H}^2 \setminus \left(\bigcup_{\tilde{\gamma}} \mathcal{N}_\epsilon(\tilde{\gamma})\right)$. So p is fixed by $\tilde{T}_{v,\epsilon}$. Hence, $\Phi_{\gamma,\epsilon}\tilde{T}_{v,\epsilon}(p) = \Phi_{\gamma,\epsilon}(p) = v = f_v(v) = f_v\Phi_{\gamma,\epsilon}(p)$.

Assume that $\mathcal{O}(v) \cap \mathcal{I}(\Phi_{\gamma,\epsilon}(p))$ is nonempty. Let $\mathcal{O}(v) \cap \mathcal{I}(\Phi_{\gamma,\epsilon}(p)) =: \{\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_m\}$. To complete the proof, we need to show that, if h_i is the right-handed generator of $\mathfrak{h}_i$, then $\tilde{T}_{v,\epsilon}(p) = h_1 \dots h_m p$. (Then, by Proposition A.1.1, we have $\Phi_{\gamma,\epsilon}T_{v,\epsilon}(p) = \Phi_{\gamma,\epsilon}(h_1 \dots h_m p) = h_1 \dots h_m \Phi_{\gamma,\epsilon}(p) = f_v\Phi_{\gamma,\epsilon}(p)$.)

We will use an induction on m which is the length of the sequence $\mathfrak{h}_1 \supsetneq \cdots \supsetneq \mathfrak{h}_m$. Let $\tilde{\gamma}_m$ be the geodesic bounding $\mathfrak{H}_m$, and let ∂_+ (∂_-, *resp.*) be the boundary component of $\mathcal{N}_\epsilon(\tilde{\gamma}_m)$ which lies on $\mathfrak{h}$ ($\mathfrak{h}^*$, *resp.*). By the induction hypothesis, for every point r of ∂_-, we have $\tilde{T}_{v,\epsilon}(r) = h_1 \dots h_{m-1} r$. Then the map $\tilde{T}' := (h_1 \dots h_{m-1})^{-1}\tilde{T}_{v,\epsilon}$ is the lift of $T_{\gamma,\epsilon}$ fixing ∂_- pointwise. So the restriction of $\tilde{T}'$ to the closure of $\mathcal{N}_\epsilon(\tilde{\gamma}_m)$ is equal to $\phi^{-1}_{\mathsf{H}_m,o,\epsilon} \circ \tilde{T} \circ \phi_{\mathsf{H}_m,o,\epsilon}$ where o is an arbitrary point in $\tilde{\gamma}_m$ and $\tilde{T}(s,t) = (s+t,t)$ for all $(s,t) \in \mathbb{R} \times [0,1]$. So, for every $p' \in \partial_+$, if $\phi_{\mathsf{H}_m,o,\epsilon}(q') = (s',1)$, then

$$\begin{aligned}\tilde{T}'(p') &= \phi^{-1}_{\mathsf{H}_m,o,\epsilon}(s+1,1) \\ &= \phi^{-1}_{\mathsf{H}_m,o,\epsilon}(\phi_{\mathsf{H}_m,o,\epsilon} h_m \phi^{-1}_{\mathsf{H}_m,o,\epsilon})(s,1) \\ &= h_m \phi^{-1}_{\mathsf{H}_m,o,\epsilon}(s,1) = h_m p'\end{aligned}$$

by Lemma A.1.2. So $\tilde{T}_{v,\epsilon}(p') = h_1 \dots h_{m-1}\tilde{T}'(p') = h_1 \dots h_{m-1} h_m p'$ for all $p' \in \partial_+$. Because p' and p belong to the same path-connected component of $\mathbb{H}^2 \setminus \left(\bigcup_{\tilde{\gamma} \in \mathcal{L}_\lambda} \mathcal{N}_\epsilon(\tilde{\gamma})\right)$, we have $\tilde{T}_{v,\epsilon}(p) = h_1 \dots h_m p$ by the path-lifting property. □

If $\tilde{F} : \mathbb{H}^2 \to \mathbb{H}^2$ is a lift of a self-homeomorphism on Σ, then it induces the automorphism on $\pi_1(\Sigma)$ defined by $g \mapsto \tilde{F} g \tilde{F}^{-1}$ for all $g \in \pi_1(\Sigma)$. It is the start of the Dehn-Nielsen-Baer theorem for closed surfaces. On the other hand, if f is a sliding permutation of some dual cube complex of finitely many simple closed geodesics, the map $g \mapsto f g f^{-1}$ is also an automorphism on $\pi_1(\Sigma)$. By the previous theorem, the automorphism induced by a sliding permutation is also realized by the conjugate action of a lift of a Dehn twist.

Corollary A.2.3. *For a simple closed geodesic γ and a vertex v of $\tilde{X}_\gamma$, and for a sufficiently small $\epsilon > 0$, the conjugate action of the sliding permutation f_v on $\pi_1(\Sigma)$ is equal to the conjugate action of $\tilde{T}_{v,\epsilon}$, that is, $f_v g f_v^{-1} = \tilde{T}_{v,\epsilon} g \tilde{T}^{-1}_{v,\epsilon}$ for all $g \in \pi_1(\Sigma)$.*

Proof. For every element g of $\pi_1(\Sigma)$, because the following diagram is commutative, there is another element $g' \in \pi_1(\Sigma)$ such that $\tilde{T}_{v,\epsilon} g \tilde{T}_{v,\epsilon}^{-1} = g'$.

$$\begin{tikzcd}
\mathbb{H}^2 \arrow[r, "\tilde{T}_{v,\epsilon}^{-1}"] \arrow[d] & \mathbb{H}^2 \arrow[r, "g"] \arrow[d] & \mathbb{H}^2 \arrow[r, "\tilde{T}_{v,\epsilon}"] \arrow[d] & \mathbb{H}^2 \arrow[d] \\
\Sigma \arrow[r, "T_{\lambda,\epsilon}^{-1}"] \arrow[rrr, dashed, bend right, "T_{\lambda,\epsilon}\circ 1 \circ T_{\lambda,\epsilon}^{-1} = 1"'] & \Sigma \arrow[r, "1"] & \Sigma \arrow[r, "T_{\lambda,\epsilon}"] & \Sigma
\end{tikzcd}$$

If u is a vertex in $\tilde{X}_\gamma$ and p is a point on $\Phi_{\gamma,\epsilon}^{-1}(u)$, then

$$\begin{aligned}
g'u &= g'\Phi_{\gamma,\epsilon}(p) = \Phi_{\gamma,\epsilon}(g'p) \\
&= \Phi_{\gamma,\epsilon}(\tilde{T}_{v,\epsilon} g \tilde{T}_{v,\epsilon}^{-1}(p)) = f_v g f_v^{-1} \Phi_{\gamma,\epsilon}(p) = f_v g f_v^{-1}(u)
\end{aligned}$$

by Theorem A.2.2 and Proposition A.1.1. Since $\pi_1(\Sigma)$ acts on $\tilde{X}_\gamma$ faithfully (by Lemma 7.1.9,) we have $\tilde{T}_{v,\epsilon} g \tilde{T}_{v,\epsilon}^{-1} = f_v g f_v^{-1}$. $\square$

Bibliography

[1] Jason Behrstock and Ruth Charney, *Divergence and quasimorphisms of right-angled Artin groups*, Math. Ann. **352** (2012), no. 2, 339–356.

[2] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, *Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups*, Geom. Topol. **21** (2017), no. 3, 1731–1804.

[3] V. N. Berestovskiĭ, *Borsuk's problem on metrization of a polyhedron*, Dokl. Akad. Nauk SSSR **268** (1983), no. 2, 273–277.

[4] M. Bestvina and M. Handel, *Train-tracks for surface homeomorphisms*, Topology **34** (1995), no. 1, 109–140.

[5] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 1–64.

[6] Mladen Bestvina and Mark Feighn, *Stable actions of groups on real trees*, Invent. Math. **121** (1995), no. 2, 287–321.

[7] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[8] Martin Robert Bridson, *Geodesics and curvature in metric simplicial complexes*, ProQuest LLC, Ann Arbor, MI, 1991, Thesis (Ph.D.)–Cornell University.

[9] Peter Buser, *The collar theorem and examples*, Manuscripta Math. **25** (1978), no. 4, 349–357.

[10] Pierre-Emmanuel Caprace and Michah Sageev, *Rank rigidity for CAT(0) cube complexes*, Geom. Funct. Anal. **21** (2011), no. 4, 851–891.

[11] Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988.

[12] Indira Chatterji, Talia Fernós, and Alessandra Iozzi, *The median class and superrigidity of actions on* CAT(0) *cube complexes*, J. Topol. **9** (2016), no. 2, 349–400, With an appendix by Pierre-Emmanuel Caprace.

[13] Matt Clay, Christopher J. Leininger, and Dan Margalit, *Abstract commensurators of right-angled Artin groups and mapping class groups*, Math. Res. Lett. **21** (2014), no. 3, 461–467.

[14] Matt T. Clay, Christopher J. Leininger, and Johanna Mangahas, *The geometry of right-angled Artin subgroups of mapping class groups*, Groups Geom. Dyn. **6** (2012), no. 2, 249–278.

[15] John Crisp and Luis Paris, *The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group*, Invent. Math. **145** (2001), no. 1, 19–36.

[16] Alex Eskin and Howard Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems **21** (2001), no. 2, 443–478.

[17] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[18] Louis Funar, *On power subgroups of mapping class groups*, J. Gökova Geom. Topol. GGT **8** (2014), 14–34.

[19] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[20] Frédéric Haglund and Frédéric Paulin, *Simplicité de groupes d'automorphismes d'espaces à courbure négative*, The Epstein birthday schrift, Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 181–248.

[21] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620.

[22] Michael Kapovich and Bernhard Leeb, *Actions of discrete groups on nonpositively curved spaces*, Math. Ann. **306** (1996), no. 2, 341–352.

[23] Sang-hyun Kim and Thomas Koberda, *Right-angled Artin groups and finite subgraphs of curve graphs*, Osaka J. Math. **53** (2016), no. 3, 705–716.

[24] Thomas Koberda, *Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups*, Geom. Funct. Anal. **22** (2012), no. 6, 1541–1590.

[25] Maxim Kontsevich and Anton Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. **153** (2003), no. 3, 631–678.

[26] Erwan Lanneau and Jean-Luc Thiffeault, *On the minimum dilatation of pseudo-Anosov homeromorphisms on surfaces of small genus*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 1, 105–144.

[27] Ian J. Leary, *A metric Kan-Thurston theorem*, J. Topol. **6** (2013), no. 1, 251–284.

[28] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.

[29] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.

[30] John W. Morgan and Peter B. Shalen, *Free actions of surface groups on* $\mathbf{R}$*-trees*, Topology **30** (1991), no. 2, 143–154.

[31] Robert C. Penner, *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 179–197.

[32] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006.

[33] Martin Roller, *Poc Sets, Median Algebras and Group Actions*, arXiv e-prints (2016), arXiv:1607.07747.

[34] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[35] Michah Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. (3) **71** (1995), no. 3, 585–617.

[36] ______, CAT(0) *cube complexes and groups*, Geometric group theory, IAS/Park City Math. Ser., vol. 21, Amer. Math. Soc., Providence, RI, 2014, pp. 7–54.

[37] Donggyun Seo, *Powers of Dehn twists generating right-angled Artin groups*, arXiv e-prints (2019), arXiv:1909.03394.

[38] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980, Translated from the French by John Stillwell.

[39] Hyunshik Shin and Balázs Strenner, *Pseudo-Anosov mapping classes not arising from Penner's construction*, Geom. Topol. **19** (2015), no. 6, 3645–3656.

[40] Richard K. Skora, *Splittings of surfaces*, J. Amer. Math. Soc. **9** (1996), no. 2, 605–616.

[41] Daniel T. Wise, *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, vol. 117, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.

Index

Symbols

$\mathbb{H}^2$	the hyperbolic plane
$d_{\mathbb{H}^2}$	the metric of the hyperbolic plane
Σ	an orientable hyperbolic surface of finite area
γ	a simple closed geodesic (on Σ)
$\tilde{\gamma}$	a lift of a simple closed geodesic γ
T_γ	the right-handed Dehn twist along a simple closed curve γ
$\mathcal{F}$	a nonempty finite set of simple closed geodesics (on Σ)
$\tilde{\mathcal{F}}$	the set of lifts of simple closed geodesics in $\mathcal{F}$
H	a halfspace of $\mathbb{H}^2$
H^*	the opposite halfspace of H
$\mathcal{H}^{\mathbb{H}^2}$	an ultrafilter of halfspaces of $\mathbb{H}^2$; or the set of halfspaces bounded by lifts of $\mathcal{F}$
$\mathcal{H}^{\mathbb{H}^2}_\gamma$	the set of halfspaces bounded by lifts of γ
$\tilde{X}_{\mathcal{F}}$	the dual cube complex of a set $\mathcal{F}$ of simple closed curves
$\tilde{X}$	a CAT(0) cube complex, or the dual cube complex equal to $\tilde{X}_{\mathcal{F}}$
d	the combinatorial metric of $\tilde{X}$
σ	the induced action of $\pi_1(\Sigma)$ on $\tilde{X}$
$\tilde{X}_\gamma$	the dual tree of a simple closed curve γ
d_γ	the combinatorial metric of $\tilde{X}_\gamma$
σ_γ	the induced action of $\pi_1(\Sigma)$ on $\tilde{X}_\gamma$
τ	the one-to-one correspondence between $\mathcal{H}^{\mathbb{H}^2} \cup \tilde{\mathcal{F}}$ and $\mathcal{H} \cup \tilde{\mathcal{H}}$

Symbols

$\Phi_{\gamma,\epsilon}$	a $\pi_1(\Sigma)$-equivariant surjective continuous map from $\mathbb{H}^2$ to $\tilde{X}_\gamma$
$\mathcal{O}(x)$	the outward orientation about a vertex x
$\mathcal{I}(x)$	the inward orientation about a vertex x
f_v	the right-handed sliding permutation centered at a vertex v

국문초록

우리는 덴 뒤틀림으로 생성된 직교 아틴군을 구했습니다. 예를 들어, 어떤 곡면에서 유한 개의 단순 폐곡선의 모든 교차 수가 1 이하일 때, 그들의 덴 뒤틀림의 일곱 제곱은 직교 아틴군을 생성합니다. 이 증명에서 우리는 단순 폐곡선의 쌍대 나무 (dual tree) 와 그 위에서의 곡면 군의 작용을 분석했습니다. 그리고 덴 뒤틀림의 올림 (lift) 에 상응하는 타원형 등거리변환 (elliptic isometry) 을 쌍대 나무에서 발견했습니다.

그리고 우리는 종수가 2 이상인 곡면에 대해 곡면군이 (동형사상으로) 충실히 (faithfully) 작용하는 어떤 CAT(0) 입방 복합체에 그 곡면군의 자기동형군이 충실히 준동형사상으로 작용할 수 있음을 증명하였습니다. 자기동형군의 작용은 곡면군의 작용에 대한 모든 정보를 포함하고 있습니다. 이를 증명하기 위해 우리는 덴 뒤틀림의 올림을 계산하기 위한 몇 가지 방법을 개발했습니다. 이 방법은 CAT(0) 입방 복합체의 여원을 갖는 부분순서집합 (pocset) 구조와 다리 (bridge) 를 이용합니다. 다리는 CAT(0) 입방 복합체의 복록 부분복합체로써, Behrstock–Charney [1] 가 연구했습니다.

주요어휘: 직교 아틴군, 사상류 군, 덴 뒤틀림, CAT(0) 초입방 복합체
학번: 2013–22906